



Configuration-Constrained Tube MPC[☆]

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ABSTRACT

This paper is about robust Model Predictive Control (MPC) for linear systems with additive and multiplicative uncertainty. A novel class of configuration-constrained polytopic robust forward invariant tubes is introduced, which admit a joint parameterization of their facets and vertices. They are the foundation for the development of novel Configuration-Constrained Tube MPC (CCTMPC) controllers that freely optimize the shape of their polytopic tube, subject to conic vertex configuration constraints, as well as associated vertex control laws by solving convex optimization problems online. It is shown that CCTMPC is—under appropriate assumptions—systematically less conservative than Rigid- and Homothetic- Tube MPC. Additionally, it is proven that there exist control systems for which CCTMPC is less conservative than Elastic Tube MPC, Disturbance Affine Feedback MPC, and Fully Parameterized Tube MPC.

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1. Introduction

During the last two decades, Tube MPC (Langson, Chrysoschoos, Raković, & Mayne, 2004) has emerged as a sensible alternative to robust dynamic programming (Bertsekas, 2012; Diehl & Bjornberg, 2004) and min–max feedback MPC (Goulart, Kerrigan, & Maciejowski, 2006) for formulating, analyzing and approximating the robust control synthesis problem (Scokaert & Mayne, 1998). It is a set-based framework (Blanchini & Miani, 2015), whose underlying principle consists of replacing system trajectories by so-called robust forward invariant tubes. These are set-valued functions in the state–space, enclosing all future states of the system, for a given feedback law, independently of the uncertainty realization (Mayne, Seron, & Raković, 2005).

Practical Tube MPC formulations rely on the parameterization of their set-valued tubes and, in the context of many existing approaches, also the feedback law. Thus, Tube MPC is harder to formulate and solve than certainty-equivalent MPC. As such, from a practical perspective, one has to ask whether the effort of investing into a robust MPC formulation pays out in the first place (Mayne, 2015). However, at least for linear systems, a variety of tractable convex Tube MPC formulations

exists. This includes the early tube MPC formulation from Chisci, Rossiter, and Zappa (2001) as well as the so-called Rigid-Tube MPC (RTMPC) (Mayne et al., 2005), Homothetic Tube MPC (HTMPC) (Langson et al., 2004; Raković, Dai, & X., 2022; Raković, Kouvaritakis, & Cannon, 2013; Raković, Kouvaritakis, Findeisen, & Cannon, 2012), Elastic Tube MPC (ETMPC) (Fleming, Kouvaritakis, & Cannon, 2015; Raković, Levine, & Açikmeşe, 2016), and Fully Parameterized Tube MPC (FPTMPC) (Raković, Kouvaritakis, Cannon, Panos, & Findeisen, 2011, 2012) formulations.

In general, it is impossible to optimize over arbitrary sets and feedback policies. Consequently, there is no unique way of answering what is the best way of implementing robust MPC. Instead, one has to trade-off between introducing conservatism and improving computational run-time performance (Köhler, Andina, Soloperto, Müller, & Allgöwer, 2019). For instance, ETMPC is less conservative than RTMPC, but comes at the cost of introducing more optimization variables. Similarly, affine feedback policies are, in general, sub-optimal (Bertsimas & Goyal, 2012), but many robust MPC schemes use affine ancillary feedback laws—or affine disturbance feedback laws as in the context of Disturbance Affine Feedback MPC (DAFMPC) (Goulart et al., 2006)—in order to arrive at a tractable reformulation. And, last but not least, the parameterization of the tube is often based on polytopes (Raković, Kouvaritakis, Findeisen, & Cannon, 2012), ellipsoids (Villanueva, Quirynen, Diehl, Chachuat, & Houska, 2017), or other classes of computer representable sets, but the choice of such set parameterizations affects the run-time and performance of the associated controller.

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Main contribution. This paper proposes a class of Configuration-Constrained Tube MPC (CCTMPC) controllers for linear discrete-time systems with additive and multiplicative uncertainty that admit a strictly contractive robust invariant set. CCTMPC is based on convex optimization and scales linearly with respect to the length of the controller's prediction horizon while its level of conservatism solely depends on the number of facet directions and the vertex configuration of the polytopes that are used to parameterize the tube. CCTMPC is—under suitable assumptions that, however, merely aim at making these controllers comparable at all—never more conservative than RTMPC and HTMPC. This statement holds for all existing variants of HTMPC, including the vertex control optimization based variants of polytopic HTMPC that are proposed in [Langson et al. \(2004, Section 3.3\)](#) and recently revisited in [Raković et al. \(2022\)](#). Here, we say that a given Tube MPC scheme “A” is never more conservative than another Tube MPC scheme “B” only if we are capable to prove that the set of all possible tubes and all possible feedback laws that are representable by scheme A contains the corresponding set of representable tubes and feedback laws of scheme B. And it is also in this rather strict sense that we shall prove that CCTMPC is never more conservative than ETMPC for systems with two states. Additionally, it is possible to show via numerical examples that CCTMPC performs better than ETMPC for a couple of selected problems in higher dimensions. Besides, a principal advantage of CCTMPC is that it can directly tackle both additive as well as multiplicative uncertainties. A similar extension is possible for homothetic tube parameterizations as briefly pointed out in [Langson et al. \(2004, Section 5\)](#) and further discussed in [Raković et al. \(2022, Section 6 B\)](#), which will be obtained as a special case of CCTMPC. In order to facilitate such extensions of ETMPC for systems with multiplicative uncertainty, however, one may need to find an affine control law that is robust with respect to the mentioned additive and multiplicative uncertainties ([Fleming et al., 2015](#)). This is in contrast to CCTMPC, which is not based on the availability of robust affine control laws. Moreover, there exist linear control systems for which CCTMPC can be proven to be less conservative and less computationally demanding than DAFMPC and FPTMPC.

Overview. This paper is organized as follows.

- Section 2 introduces the problem formulation.
- Section 3 reviews the definition of template polyhedra. Novel conic configuration domains are introduced, which correspond to the set of parameters of template polyhedra that share a given face configuration. [Theorem 1](#) provides a unique characterization of these configuration domains. Moreover, [Theorem 2](#) elaborates on their use for computing vertex representations of polytopes that are given in half-space representation.
- Section 4 presents an exact and computationally tractable condition under which a sequence of configuration-constrained polytopes form a robust forward invariant tube; see [Corollary 4](#).
- Section 5 discusses the practical implementation of CCTMPC, recursive feasibility, and asymptotic stability; see [Theorem 4](#).
- Section 6 discusses three tutorial examples, supported by numerical illustrations. It is explained and visualized under which assumptions CCTMPC is less conservative than RTMPC, HTMPC, and ETMPC. An explicit example for an uncertain linear system with 4 states and 1 control input is constructed, for which CCTMPC is less conservative than both DAFMPC and FPTMPC.
- Finally, Section 7 concludes the paper.

2. Tube MPC

This section introduces the basic notation for uncertain linear systems, reviews the concept of robust forward invariance, and discusses our problem formulation.

2.1. Uncertain linear systems

This paper is concerned with systems of the form

$$x_{k+1} = A_k x_k + B_k u_k + C w_k. \quad (1)$$

Here, $x_k \in \mathbb{X}$ denotes the state at time $k \in \mathbb{N}$ and $u_k \in \mathbb{U}$ the control input. The state and control constraint sets, $\mathbb{X} \subseteq \mathbb{R}^{n_x}$ and $\mathbb{U} \subseteq \mathbb{R}^{n_u}$, are assumed to be closed and convex. In the most general setting, the matrices

$$[A_k, B_k] \in \Delta \subseteq \mathbb{R}^{n_x \times (n_x + n_u)}$$

and vectors $w_k \in \mathbb{W}$ are both unknown. The polytope

$$\Delta \stackrel{\text{def}}{=} \text{conv}([\bar{A}_1, \bar{B}_1], [\bar{A}_2, \bar{B}_2], \dots, [\bar{A}_l, \bar{B}_l])$$

has given vertices, $[\bar{A}_1, \bar{B}_1], [\bar{A}_2, \bar{B}_2], \dots, [\bar{A}_l, \bar{B}_l]$. The set $\mathbb{W} \subseteq \mathbb{R}^{n_w}$ is assumed to be compact. Additionally, $C \in \mathbb{R}^{n_x \times n_w}$ is assumed to be given and constant.

2.2. Robust forward invariant tubes

Let $\mathcal{U} = \{\mu \mid \mu : \mathbb{R}^{n_x} \rightarrow \mathbb{U}\}$ denote the set of control laws. In the following, we use the notation

$$f(X, \mu) \stackrel{\text{def}}{=} \left\{ Ax + B\mu(x) + Cw \mid \begin{array}{l} x \in X, w \in \mathbb{W} \\ [A, B] \in \Delta \end{array} \right\}$$

to denote the set propagation function. It is defined for all $X \subseteq \mathbb{R}^{n_x}$ and all control laws $\mu \in \mathcal{U}$.

Definition 1. A set $X_s \subseteq \mathbb{R}^{n_x}$ is called a robust control invariant (RCI) set of (1) if there exists a $\mu_s \in \mathcal{U}$ for which $X_s \supseteq f(X_s, \mu_s)$.

Definition 2. A sequence of sets $X_0, X_1, \dots, X_N \subseteq \mathbb{R}^{n_x}$, with $N \in \mathbb{N} \cup \{\infty\}$, is called a robust forward invariant tube of (1) if there exists a sequence of control laws $\mu_0, \mu_1, \dots, \mu_N \in \mathcal{U}$ such that $X_{k+1} \supseteq f(X_k, \mu_k)$ for all indices $k \in \{0, 1, \dots, N-1\}$.

The function f should not be mixed up with the function

$$F(X) \stackrel{\text{def}}{=} \left\{ X^+ \subseteq \mathbb{R}^{n_x} \mid \begin{array}{l} \forall x \in X, \exists u \in \mathbb{U} : \\ \forall [A, B] \in \Delta, \forall w \in \mathbb{W}, \\ Ax + Bu + Cw \in X^+ \end{array} \right\},$$

which is also defined for all sets $X \subseteq \mathbb{R}^{n_x}$, but does not depend on μ . The functions f and F are, however, closely related, since for two given sets $X, X^+ \subseteq \mathbb{R}^{n_x}$ we have

$$\exists \mu \in \mathcal{U} : X^+ \supseteq f(X, \mu) \iff X^+ \in F(X).$$

Note that while f is set-valued, the values of F are sets-of-sets. Despite this apparent complication, working with F is sometimes more elegant than working with f . For instance, instead of [Definition 1](#), one could also say that $X_s \subseteq \mathbb{R}^{n_x}$ is an RCI set if $X_s \in F(X_s)$. This notation highlights that a direct parameterization of control laws can eventually be avoided as long as one is able to find a tractable representation of F .

2.3. Tube MPC

This paper is focused on formulating, analyzing, and solving Tube MPC problems of the form

$$\begin{aligned} \min_{X, \mu} \quad & L_0(X_0) + \sum_{k=0}^{N-1} L(X_k, \mu_k) + L_N(X_N) \\ \text{s.t.} \quad & \begin{cases} \forall k \in \{0, 1, \dots, N-1\}, \\ X_{k+1} \supseteq f(X_k, \mu_k), \hat{x} \in X_0, X_k \subseteq \mathbb{X}, \\ X_N \subseteq \mathbb{X}, X_k \in \mathcal{X}, X_N \in \mathcal{X}, \mu_k \in \mathcal{U}. \end{cases} \end{aligned} \quad (2)$$

In the context of this paper, $\mathcal{X}$ denotes a suitable class of polytopes—to be specified in Section 5. This means that the set-valued optimization variables of (2), $X_0, X_1, \dots, X_N$, form a robust forward invariant polytopic tube satisfying the state constraints, $X_k \subseteq \mathbb{X}$ over the prediction horizon of the controller. In this context, $\hat{x}$ denotes the current state measurement and the constraint $\hat{x} \in X_0$ ensures that the first set of our optimized tube contains $\hat{x}$. Note that the control laws $\mu = (\mu_0, \mu_1, \dots, \mu_{N-1})$ are freely optimized: we shall not impose any restrictions on these control laws apart from requiring that they respect all constraints in (2).

A discussion on how to design the stage cost $L : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}$ as well as the initial- and end-cost $L_0 : \mathcal{X} \rightarrow \mathbb{R}$ and $L_N : \mathcal{X} \rightarrow \mathbb{R}$ can be found in Section 5.

3. Template polyhedra

Our construction of a tractable reformulation of (2) is based on parametric polyhedra, which have been analyzed by many authors (Bemporad, Morari, Dua, & Pistikopoulos, 2002; Blanchini & Miani, 2015; Mayne et al., 2005; Raković, Kouvaritakis, Findeisen, & Cannon, 2012; Raković et al., 2016). Therefore, our introduction of polyhedra in Section 3.1 is kept short, but Sections 3.2 and 3.3 review how generalized faces and facial configurations of template polyhedra can be defined and analyzed. Novel ideas are presented in Sections 3.4 and 3.5, which introduce a class of configuration-constrained polytopes that admit a joint parameterization of their facets and vertices; see Theorem 2.

3.1. Polyhedra and polytopes

Let $Y \in \mathbb{R}^{m \times n}$ be a given matrix. Throughout this paper, polyhedra with parameter $y \in \mathbb{R}^m$ are denoted by

$$P(Y, y) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid Yx \leq y\} \subseteq \mathbb{R}^n.$$

Every polyhedron is both convex and closed. If $P(Y, y)$ is also bounded, it is called a polytope (Bitsoris, 1988; Blanchini & Miani, 2015).

Definition 3. The set of feasible parameters,

$$\mathbb{Y} \stackrel{\text{def}}{=} \{y \in \mathbb{R}^m \mid P(Y, y) \neq \emptyset\},$$

is called the natural domain of the template $Y \in \mathbb{R}^{m \times n}$.

Proposition 1. We have $\mathbb{R}_+^m \subseteq \mathbb{Y}$, where $\mathbb{R}_+^m$ denotes the set of componentwise non-negative vectors in $\mathbb{R}^m$.

Proof. If we have $y \in \mathbb{R}_+^m$, then the inequality $Yx \leq y$ is trivially satisfied for $x = 0$. Consequently, $P(Y, y)$ is non-empty for all $y \in \mathbb{R}_+^m$, which implies $\mathbb{R}_+^m \subseteq \mathbb{Y}$. $\square$

3.2. Faces of polyhedra

Since the developments in this paper build upon understanding the geometry of polyhedra, this section briefly recalls how faces of polyhedra are defined. Let

$$I = \{i_1, i_2, \dots, i_{|I|}\} \subseteq \{1, 2, \dots, m\}$$

be an index set. In the following, the notation

$$Y_I \stackrel{\text{def}}{=} \begin{bmatrix} Y_{i_1}^\top, Y_{i_2}^\top, \dots, Y_{i_{|I|}}^\top \end{bmatrix}^\top$$

$$\text{and } y_I \stackrel{\text{def}}{=} \begin{bmatrix} y_{i_1}, y_{i_2}, \dots, y_{i_{|I|}} \end{bmatrix}^\top$$

is used to denote the matrix (or vector) constructed by collecting all the rows of Y (or coefficients of y), whose index is in the set I . Moreover,

$$\mathcal{F}_I(y) \stackrel{\text{def}}{=} \{x \in P(Y, y) \mid Y_I x \geq y_I\} \quad (3)$$

denotes the face of $P(Y, y)$ associated to I . This notation is formally also defined for the case $I = \emptyset$: $\mathcal{F}_\emptyset(y) = P(Y, y)$. Additionally, we recall that 0-dimensional faces are called vertices, 1-dimensional faces are called edges, and faces with codimension 1 are called facets. The intersection of faces is again a face, since

$$\forall I, J \subseteq \{1, 2, \dots, m\}, \quad \mathcal{F}_I(y) \cap \mathcal{F}_J(y) = \mathcal{F}_{I \cup J}(y).$$

Moreover, every face of a polyhedron is a polyhedron. Similarly, the faces of a polytope are also polytopes.

Definition 4. The polyhedron $P(Y, y)$ is called entirely simple if all of its non-empty faces, $\mathcal{F}_I(y) \neq \emptyset$, satisfy $\text{rank}(Y_I) = |I|$.

Note that n -dimensional entirely simple polyhedra satisfy that all of their non-empty faces $\mathcal{F}_I(y)$ satisfy $|I| \leq n$, since n is an upper bound on the rank of $Y \in \mathbb{R}^{m \times n}$.

Remark 1. An n -dimensional polytope (with $n \geq 1$) is called simple, if all its vertices lie in exactly n edges, or equivalently, if it is dual to a simplicial polytope (Grünbaum, 1967). This naming convention is consistent with Definition 4 in the sense that entirely simple polytopes are also simple. Note, however, that Definition 4 depends on the algebraic parameterization of the polyhedron, not only on its geometric image.

3.3. Face configurations

A face configuration of a polyhedron is the collection of index sets of its non-empty faces.

Definition 5. The face configuration $\mathcal{C}(y)$ of $P(Y, y)$ is

$$\mathcal{C}(y) \stackrel{\text{def}}{=} \{I \subseteq \{1, \dots, m\} \mid \mathcal{F}_I(y) \neq \emptyset\}.$$

Moreover, a set $\mathcal{I} \subseteq 2^{\{1, 2, \dots, m\}}$ is called a partial face configuration of the polyhedron $P(Y, y)$, if $\mathcal{I} \subseteq \mathcal{C}(y)$.

We say that two polyhedra, $P(Y, y)$ and $P(Y, y')$, have the same configuration if $\mathcal{C}(y) = \mathcal{C}(y')$. Similarly, we say that the polyhedra $P(Y, y)$ and $P(Y, y')$ share the partial face configuration $\mathcal{I}$ whenever

$$\mathcal{C}(y) \cap \mathcal{I} = \mathcal{C}(y') \cap \mathcal{I}.$$

At this point, it is recommended to study Fig. 1. It shows a pair of polytopes whose face configurations do not coincide. The left polytope is not entirely simple, since the face $\mathcal{F}_I(y)$ with $I = \{1, 2, 3, 4\}$ is non-empty—the corresponding matrix $Y_I \in \mathbb{R}^{4 \times 3}$ cannot possibly have rank 4. In contrast to this, the polytope $P(Y, y')$ in the middle of Fig. 1 is an entirely simple polytope.

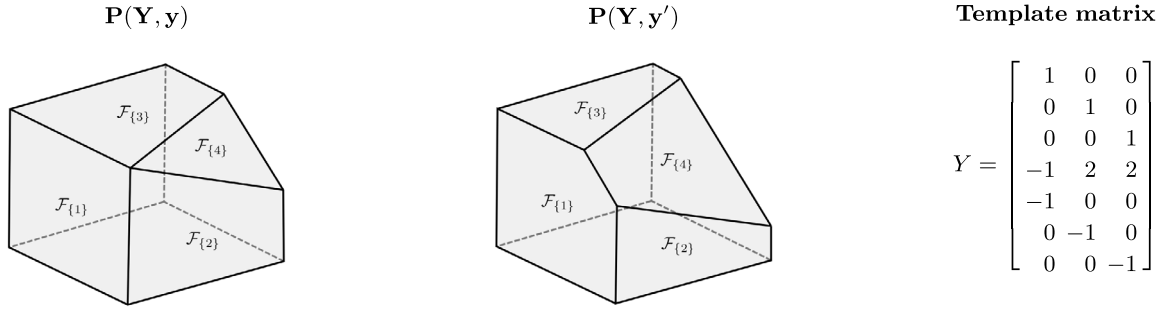


Fig. 1. The polytopes $P(Y, y)$ and $P(Y, y')$ with $y = [1, 1, 1, 3, 1, 1, 1]^T$ and $y' = [1, 1, 1, 5/2, 1, 1, 1]^T$ for a given template matrix $Y \in \mathbb{R}^{7 \times 3}$. Note that both polytopes have 7 two-dimensional facets; that is, $\mathcal{I} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}\}$ is a shared partial face configuration. However, $P(Y, y)$ has 9 vertices, while $P(Y, y')$ has 10 vertices. This implies that $P(Y, y)$ and $P(Y, y')$ do not have the same face configuration, $\mathcal{C}(y) \neq \mathcal{C}(y')$.

Since there are only finitely many configurations, $\mathcal{C}$ must be a piecewise constant function on $\mathbb{Y}$. As such, it is interesting to ask which polyhedra $P(Y, y)$ have a locally stable configuration, such that $\mathcal{C}(y) = \mathcal{C}(y')$ holds for all y' in a small open neighborhood of y . As it turns out, such a local stability property holds if and only if $P(Y, y)$ is an entirely simple polyhedron. This statement will be obtained as a special case of a more powerful global stability statement—see [Theorem 1](#) and [Corollary 2](#).

3.4. Configuration domains

In general it is hard to bound the number of faces of high dimensional polyhedra—let alone to compute or to classify them ([McMullen, 1971](#); [Stanley, 1980](#)). Instead, this paper attempts to characterize the set of polyhedra that share a given (partial) face configuration.

Definition 6. Let $\mathcal{I}$ be a given collection of subsets of the index set $\{1, 2, \dots, m\}$. The set

$$\mathbb{Y}_{\mathcal{I}} \stackrel{\text{def}}{=} \{y \in \mathbb{R}^m \mid \forall I \in \mathcal{I}, \mathcal{F}_I(y) \neq \emptyset\}.$$

is called the configuration domain of $\mathcal{I}$.

An important observation is that configuration domains admit computationally tractable representations.

Lemma 1. The configuration domain $\mathbb{Y}_{\mathcal{I}}$ is for any given set $\mathcal{I} \subseteq 2^{\{1, 2, \dots, m\}}$ a polyhedral cone.

Proof. Let us introduce the auxiliary sets

$$\mathbb{K}_I \stackrel{\text{def}}{=} \{[x^T, y^T]^T \mid Yx \leq y, Y_I x \geq y_I\} \quad (4)$$

for all $I \in \mathcal{I}$, which are, by construction, polyhedral cones in $\mathbb{R}^{n+m}$, since both constraints in (4) are linear (and, thus, homogeneous) in the stacked vector $[x^T, y^T]^T$. Since the face $\mathcal{F}_I(y)$ can be written in the form

$$\mathcal{F}_I(y) = \{x \in P(Y, y) \mid Y_I x \geq y_I\},$$

the set $\mathbb{Y}_{\{I\}} = \{y \mid \mathcal{F}_I(y) \neq \emptyset\}$ is the projection of $\mathbb{K}_I$ onto the last m coordinates; that is, $\mathbb{Y}_{\{I\}} = [0 \ 1] \cdot \mathbb{K}_I$. Since projections preserve polyhedral conic structures, $\mathbb{Y}_{\{I\}}$ is a polyhedral cone. Furthermore, since finite intersections also preserve polyhedral conic structures,

$$\mathbb{Y}_{\mathcal{I}} = \bigcap_{I \in \mathcal{I}} \mathbb{Y}_{\{I\}}$$

is a polyhedral cone, which completes our proof. $\square$

Corollary 1. The set $\mathbb{Y}$ is a convex polyhedral cone with non-empty interior in $\mathbb{R}^m$.

Proof. Since $\mathbb{Y} = \mathbb{Y}_{\{\emptyset\}}$, the fact that $\mathbb{Y}$ is a polyhedral cone is a special case of [Lemma 1](#). Next, [Proposition 1](#) states that $\mathbb{Y}$ contains the open set $\mathbb{R}_{++}^m$, implying that $\mathbb{Y}$ has a non-empty interior. $\square$

In contrast to $\mathbb{Y}$, the configuration domains $\mathbb{Y}_{\mathcal{I}}$ do not necessarily have a non-empty interior. Therefore, the definition below introduces a notion of regularity for $\mathbb{Y}_{\mathcal{I}}$.

Definition 7. A configuration domain $\mathbb{Y}_{\mathcal{I}}$ is regular if its interior in $\mathbb{R}^m$ is non-empty.

The theorem below provides a unique characterization of regular configuration domains.

Theorem 1. Let $\mathcal{I}$ be a set of subsets of $\{1, \dots, m\}$. The following statements are equivalent:

- (1) The set $\mathbb{Y}_{\mathcal{I}}$ is a regular configuration domain.
- (2) There exists a point $\sigma \in \mathbb{Y}_{\mathcal{I}}$ such that $P(Y, \sigma)$ is an entirely simple polyhedron.

[Theorem 1](#) can be interpreted as a global configuration stability result in the sense that it contains the following local stability result as a special case.

Corollary 2. The polyhedron $P(Y, y)$ is locally configuration stable if and only if it is entirely simple.

Moreover, another important consequence of [Theorem 1](#) is that entirely simple polyhedra are prevalent.

Corollary 3. The polyhedron $P(Y, y)$ is entirely simple for almost all $y \in \mathbb{Y}$.

Proofs of [Theorem 1](#), [Corollary 2](#), and [Corollary 3](#) can be found in [Appendices A](#) and [B](#).

3.5. Vertex configuration domains of polytopes

Let $\sigma \in \mathbb{R}^m$ be such that $P(Y, \sigma)$ is an entirely simple polytope. We use the notation $\mathcal{I}_n$ to denote the set of subsets of $\{1, 2, \dots, m\}$ with n elements such that

$$\mathcal{V} \stackrel{\text{def}}{=} \{I \in \mathcal{I}_n \mid \mathcal{F}_I(\sigma) \neq \emptyset\}$$

can be interpreted as the index set collection associated with the vertices of $P(Y, \sigma)$. Clearly, if $\bar{m}$ denotes the number of vertices of $P(Y, \sigma)$, we can enumerate them as

$$\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_{\bar{m}}\} \text{ and define } V_i \stackrel{\text{def}}{=} Y_{\mathcal{V}_i}^{-1} \mathbb{1}_{\mathcal{V}_i},$$

with $\mathbb{1}$ being the unit matrix and $\mathbb{1}_{\mathcal{V}_i}$ a matrix whose n rows are the n unit vectors of $\mathbb{R}^m$ with coordinate index in $\mathcal{V}_i$. The matrices

Y_{V_i} are invertible, since $P(Y, \sigma)$ is assumed to be entirely simple. The points

$$V_1\sigma, V_2\sigma, \dots, V_{\bar{m}}\sigma \in P(Y, \sigma)$$

correspond to the $\bar{m}$ isolated vertices of $P(Y, \sigma)$. Next, for any given $y \in \mathbb{R}^m$ an equivalence of the form

$$\begin{aligned} \mathcal{F}_{V_i}(y) \neq \emptyset &\iff V_i y \in P(Y, y) \\ &\iff YV_i y \leq y \\ &\iff (YV_i - \mathbb{1})y \leq 0 \end{aligned} \quad (5)$$

holds. The latter inequality motivates the introduction of the conic constraint matrix

$$E \stackrel{\text{def}}{=} \begin{pmatrix} YV_1 - \mathbb{1} \\ YV_2 - \mathbb{1} \\ \vdots \\ YV_{\bar{m}} - \mathbb{1} \end{pmatrix} \quad (6)$$

such that we can write the vertex configuration domain $\mathbb{Y}_V$ in its explicit form

$$\mathbb{Y}_V \stackrel{(5), (6)}{=} \{y \in \mathbb{R}^m \mid Ey \leq 0\}. \quad (7)$$

The role of $\mathbb{Y}_V$ in the ongoing developments of this paper is clarified by the following theorem. It implies that for all $y \in \mathbb{Y}_V$, the polytope $P(Y, y)$ has at most $\bar{m}$ vertices.

Theorem 2. Let $P(Y, \sigma)$ be an entirely simple polytope with vertex configuration domain $\mathbb{Y}_V$, as defined in (7). Then, we have

$$\forall y \in \mathbb{Y}_V, \quad P(Y, y) = \text{conv}(V_1 y, V_2 y, \dots, V_{\bar{m}} y),$$

where $\text{conv}(\cdot)$ denotes the convex hull operator.

Proof. Note that for the scalar case, $n = 1$, we may assume $Y = (-1, 1)^T$ without loss of generality; that is

$$P(Y, y) = [-y_1, y_2]$$

is an interval. The matrices $V_1 = (-1, 0)$ and $V_2 = (0, 1)$ locate the two vertices, $V_1 y = -y_1$ and $V_2 y = y_2$. The associated vertex configuration domain is then given by

$$\mathbb{Y}_V = \{y \in \mathbb{R}^2 \mid y_1 + y_2 \geq 0\}.$$

Thus, for $n = 1$, the statement of the theorem holds, as $P(Y, y) = [-y_1, y_2] = \text{conv}(-y_1, y_2)$ holds for all $y \in \mathbb{Y}_V$.

Next, we may assume $n \geq 2$. By the definition of $\mathbb{Y}_V$ the sets $\{V_i y\}$ are non-empty faces of $P(Y, y)$; that is, $V_i y \in P(Y, y)$ for all $i \in \{1, 2, \dots, \bar{m}\}$, compare (5). Hence, since $P(Y, y)$ is convex, we have the inclusion

$$\text{conv}(V_1 y, V_2 y, \dots, V_{\bar{m}} y) \subseteq P(Y, y). \quad (8)$$

For the reverse inclusion, a global geometric argument is needed. Let S^{n-1} be the unit sphere in $\mathbb{R}^n$. Since $n \geq 2$, the surface area of S^{n-1} is well-defined and given by¹

$$|S^{n-1}| = \frac{2\pi^{\frac{n}{2}}}{\Gamma(n/2)}.$$

Moreover, for any given point $x \in P(Y, y)$, let

$$\mathcal{N}(x, y) \stackrel{\text{def}}{=} \left\{ z \in S^{n-1} \mid \begin{array}{l} \forall x' \in P(Y, y), \\ z^T(x' - x) \leq 0 \end{array} \right\}$$

denote the intersection of S^{n-1} and the normal cone of $P(Y, y)$ at x . The Gauss–Federer curvature (Federer, 1959) of any vertex v of $P(Y, y)$ can now be defined as

$$\kappa(v, y) \stackrel{\text{def}}{=} |\mathcal{N}(v, y)|,$$

where $|\mathcal{N}(v, y)|$ denotes the surface area of the spherical polygon $\mathcal{N}(v, y)$. Next, the key observation of this proof is that κ can be used as an invariant: if $P(Y, y)$ is an entirely simple polytope with $y \in \mathbb{Y}_V$, then we have

$$\kappa(V_i y, y) = \kappa(V_i \sigma, \sigma). \quad (9)$$

This follows as the Gauss–Federer curvature of isolated vertices only depends on the direction of the normal vectors of the facets intersecting at this vertex. Moreover,

$$\sum_{i=1}^{\bar{m}} \kappa(V_i \sigma, \sigma) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(n/2)}, \quad (10)$$

holds because $P(Y, \sigma)$ has exactly $\bar{m}$ vertices, namely, all $V_i \sigma$ for $i \in \{1, 2, \dots, \bar{m}\}$. Eq. (9) can be interpreted as a special case of the Gauss–Bonnet theorem for convex polytopes (Federer, 1959), which, in our case, simply follows from the fact that the spherical polygons $\mathcal{N}(V_i \sigma, \sigma)$ of the vertices of $P(Y, \sigma)$ form a complete partition of the unit sphere. Substituting (9) in (10), we find that

$$\sum_{i=1}^{\bar{m}} \kappa(V_i y, y) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(n/2)}, \quad (11)$$

still assuming that $P(Y, y)$ is an entirely simple polytope. Suppose $P(Y, y)$ had $\bar{m} + k$ vertices with $k > 0$, where

$$v_1^*, v_2^*, \dots, v_k^* \notin \text{conv}(V_1 y, V_2 y, \dots, V_{\bar{m}} y)$$

denote the vertices of $P(Y, y)$ that are not in the convex hull of the $\bar{m}$ known vertices of the form $V_i y$. If this was the case, then we would have

$$\sum_{i=1}^{\bar{m}} \kappa(V_i y, y) + \sum_{i=1}^k \kappa(v_i^*, y) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(n/2)},$$

which, in turn, would contradict (11), since we have $\kappa(v_i^*, y) > 0$. Consequently, all vertices of $P(Y, y)$ are contained in the convex hull of the points $V_i y$, which—due to (8)—implies that

$$P(Y, y) = \text{conv}(V_1 y, V_2 y, \dots, V_{\bar{m}} y). \quad (12)$$

Last but not least, by Corollary 3, $P(Y, y)$ is entirely simple for almost all $y \in \mathbb{Y}_V$. Thus, so far, we have shown that (12) holds for almost all $y \in \mathbb{Y}_V$. However, since the set of y for which (12) holds is closed in $\mathbb{R}^m$, it follows that the statement of this theorem holds for all $y \in \mathbb{Y}_V$. $\square$

Remark 2. The computational complexity of the above construction depends on the number of facet normals m and vertices $\bar{m}$ of $P(Y, \sigma)$. In this context, we would like to emphasize the following observations:

- (1) m and $\bar{m}$ only depend on Y and σ , which are chosen by us. This allows a trade-off between the accuracy and complexity of the set representation.
- (2) E is typically sparse. Its number of non-zero entries is much smaller than $\bar{m} \cdot m^2$. Moreover, E often has a significant number of redundant rows.

Remark 3. In $\mathbb{R}^2$, one may assume—without loss of generality—that the rows of the matrix Y have the form

$$Y_i = [\cos(\varphi_i), \sin(\varphi_i)] \in \mathbb{R}^{1 \times 2}$$

with $0 \leq \varphi_1 < \varphi_2 < \dots < \varphi_m < 2\pi$. Note that one can always normalize the constraints $Yx \leq y$ row-wise and sort them with respect to their argument in polar coordinates. Moreover, we may assume that $m \geq 3$ and $\varphi_{i+1} - \varphi_i < \pi$ and $\varphi_m - \varphi_1 > \pi$ for all $i \in \{1, \dots, m-1\}$, such that $P(Y, y)$ is for all $y \in \mathbb{Y}$ a polytope.

¹ We use the notation $\Gamma(n) \stackrel{\text{def}}{=} \int_0^\infty e^{-t} t^{n-1} dt$ for any $n \geq 1$.

The initial parameter is chosen as $\sigma = [1, 1, \dots, 1]^T \in \mathbb{R}^m$ and the vertices enumerated as

$$\mathcal{V}_i = \{i, i+1\} \quad \text{and} \quad \mathcal{V}_m = \{m, 1\}$$

for all $i \in \{1, \dots, m-1\}$ and $\mathcal{V} = \{\mathcal{V}_1, \dots, \mathcal{V}_m\}$. Note that this construction is such that the auxiliary points $\xi_i = Y_i^T$ satisfy, for all $i, j \in \{1, \dots, m\}$ with $i \neq j$,

$$Y_i \xi_i = \cos(\varphi_i)^2 + \sin(\varphi_i)^2 = 1 = \sigma_i,$$

$$\text{and} \quad Y_j \xi_i = \cos(\varphi_i - \varphi_j) < 1 = \sigma_j.$$

Consequently, we have $\xi_i \in \mathcal{F}_{\{i\}}$ but $\xi_i \notin \mathcal{F}_{\{j\}}$ for $i \neq j$. This is sufficient to ensure that $P(Y, \sigma)$ is entirely simple and has $\bar{m} = m$ isolated vertices.

We claim that there exists for every non-empty polytope $P(Y, y') \subset \mathbb{R}^2$ a $y \in \mathbb{R}^m$ such that $P(Y, y') = P(Y, y)$ and $Ey \leq 0$. The argument is as follows. If we set

$$\forall i \in \{1, \dots, m\}, \quad y_i \stackrel{\text{def}}{=} \max_{x \in P(Y, y')} Y_i x, \quad (13)$$

then $P(Y, y') = P(Y, y)$ holds by construction. Also, by (13), all facets of $P(Y, y)$ are non-empty. But this is only possible, if the intersections of all neighboring facets are contained in $P(Y, y)$, that is

$$\forall i \in \{1, \dots, m\}, \quad V_i y \in P(Y, y).$$

By (6), this implies $Ey \leq 0$. Thus, our claim holds.

4. Configuration-constrained polytopic tubes

This section presents the construction of configuration-constrained polytopic robust forward invariant tubes. As before, $\mathbb{Y}_{\mathcal{V}}$ denotes the vertex configuration domain of a given entirely simple polytope $P(Y, \sigma)$, see (7); and E is constructed as in (6). A sequence of polytopes with parameters $y_0, y_1, \dots \in \mathbb{Y}_{\mathcal{V}}$ is robust forward invariant if

$$\forall k \in \mathbb{N}, \quad P(Y, y_{k+1}) \in F(P(Y, y_k)). \quad (14)$$

We recall that $\mathbb{U}$ is assumed to be closed and convex while $\mathbb{W}$ is compact. We additionally define

$$\bar{w}_i \stackrel{\text{def}}{=} \max_{w \in \mathbb{W}} Y_i C w$$

such that $P(Y, \bar{w})$ is a tight polytopic enclosure of $C\mathbb{W}$.

4.1. Vertex control laws

A well-known necessary and sufficient condition for ensuring that (14) holds is that there exist for every vertex $v_{k,i}$ of $P(Y, y_k)$ a control input $u_{k,i} \in \mathbb{U}$ such that

$$\bar{A}_j v_{k,i} + \bar{B}_j u_{k,i} + C w \in P(Y, y_{k+1})$$

holds for all $w \in \mathbb{W}$ and all vertices of Δ —that is, for all $j \in \{1, \dots, l\}$. This statement about the existence of vertex control laws has—in a slightly different version—been proven by Gutman and Cwikel (1986). In order to briefly discuss the role of this result in the context of the current manuscript, we introduce the convex set

$$\mathcal{F} \stackrel{\text{def}}{=} \left\{ (y, y^+) \in \mathbb{Y}_{\mathcal{V}}^2 \mid \begin{array}{l} \exists u_1, u_2, \dots, u_{\bar{m}} \in \mathbb{U} : \\ \forall i \in \{1, \dots, \bar{m}\}, j \in \{1, \dots, l\}, \\ Y \bar{A}_j V_i y + Y \bar{B}_j u_i + \bar{w} \leq y^+ \end{array} \right\}.$$

For the sake of the completeness, we provide a short proof of the following corollary, although its statement is essentially a direct consequence of Theorem 2, the original results from Gutman and Cwikel (1986), and certain properties of linear systems with multiplicative uncertainties (Kothare, Balakrishnan, & Morari, 1996).

Corollary 4. The configuration-constrained polytopes $P(Y, y)$ and $P(Y, y^+)$ with $y, y^+ \in \mathbb{Y}_{\mathcal{V}}$ satisfy

$$P(Y, y^+) \in F(P(Y, y)) \iff (y, y^+) \in \mathcal{F}. \quad (15)$$

Proof. Our proof is divided into two parts. The first part shows that $(y, y^+) \in \mathcal{F}$ implies (15) and the second part establishes the reverse implication.

Part I. Let us assume that $(y, y^+) \in \mathcal{F}$. The definition of $\mathcal{F}$ implies that $y \in \mathbb{Y}_{\mathcal{V}}$ and, consequently, Theorem 2 implies that every $x \in P(Y, y)$ can be written in the form

$$x = \sum_{i=1}^{\bar{m}} \theta_i(x) V_i y$$

for suitable scalars $\theta_1(x), \theta_2(x), \dots, \theta_{\bar{m}}(x) \geq 0$ satisfying $\sum_{i=1}^{\bar{m}} \theta_i(x) = 1$. The definition of $\mathcal{F}$ implies that there exists for every vertex $V_i y$ a control input $u_i \in \mathbb{U}$ with

$$Y \bar{A}_j V_i y + Y \bar{B}_j u_i + \bar{w} \leq y^+ \quad (16)$$

for all $j \in \{1, \dots, l\}$. The corresponding control law

$$\mu(x) \stackrel{\text{def}}{=} \sum_{i=1}^{\bar{m}} \theta_i(x) u_i \quad (17)$$

satisfies $\mu(x) \in \mathbb{U}$, since $\mathbb{U}$ is convex. Next, for any given $x \in P(Y, y)$, we can multiply (16) with $\theta_i(x)$ on both sides and take the sum over i in order to show that

$$\forall j \in \{0, \dots, l\}, \quad Y (\bar{A}_j x + \bar{B}_j \mu(x)) + \bar{w} \leq y^+. \quad (18)$$

Moreover, for the same given $x \in P(Y, y)$, we have

$$\forall w \in \mathbb{W}, \quad \forall [A, B] \in \Delta, \quad Ax + B\mu(x) + Cw \in P(Y, y^+)$$

$$\iff \begin{cases} \forall i \in \{1, \dots, m\}, \\ \max_{w \in \mathbb{W}} \max_{[A, B] \in \Delta} Y_i (Ax + B\mu(x) + Cw) \leq y_i^+ \end{cases}$$

$$\iff \begin{cases} \forall i \in \{1, \dots, m\}, \\ \max_{[A, B] \in \Delta} Y_i (Ax + B\mu(x)) + \bar{w}_i \leq y_i^+ \end{cases}$$

$$\iff \begin{cases} \forall j \in \{0, \dots, l\} \\ Y (\bar{A}_j x + \bar{B}_j \mu(x)) + \bar{w} \leq y^+ \end{cases}.$$

The latter equivalence and (18) imply that (15) holds.

Part II. Reversely, if (15) holds for given $y, y^+ \in \mathbb{Y}_{\mathcal{V}}$, there exists for every point $V_i y \in P(Y, y)$ a control input $u_i \in \mathbb{U}$ such that for all $[A, B] \in \Delta$ we have

$$\begin{aligned} & \forall w \in \mathbb{W}, \quad AV_i y + Bu_i + Cw \in P(Y, y^+) \\ \Leftrightarrow & \forall j \in \{1, \dots, m\}, \quad \max_{w \in \mathbb{W}} Y_j (AV_i y + Bu_i + Cw) \leq y_j^+ \\ \Leftrightarrow & YAV_i y + YBu_i + \bar{w} \leq y^+. \end{aligned}$$

In particular, this equivalence holds at all vertices of Δ . This is sufficient to conclude that $(y, y^+) \in \mathcal{F}$. $\square$

Remark 4. If $P(Y, y)$ is a simplex with $\bar{m} = n_x + 1$ isolated vertices, one can interpolate the vertex control inputs $u_1, u_2, \dots, u_{\bar{m}}$ by an affine control law of the form $\mu(x) = Kx + k$. Here, $K \in \mathbb{R}^{n_u \times n_x}$ and $k \in \mathbb{R}^{n_u}$ can be found by solving the linear equation system

$$\forall i \in \{1, \dots, \bar{m}\}, \quad KV_i y + k = u_i. \quad (19)$$

However, if $P(Y, y)$ has $\bar{m} > n_x + 1$ isolated vertices, System (19) is, in general, overdetermined. This observation is exploited in Section 6 to explain why CCTMPC is potentially less conservative than Tube MPC schemes that use affine feedback laws.

4.2. Contractivity of polytopic tubes

Corollary 4 provides a tractable convex reformulation of Condition (14). However, developing practical Tube MPC schemes requires additional assumptions.

Assumption 1. The polytope $P(Y, \sigma)$ is entirely simple, satisfies $0 \in P(Y, \sigma) \subseteq \mathbb{X}$, and is β -contractive,

$$\beta \cdot P(Y, \sigma) \in F(P(Y, \sigma)) \quad \text{with} \quad \beta \in [0, 1).$$

Assuming that a control system admits a compact, convex, and β -contractive set $\mathcal{C}$ with $0 \in \text{int}(\mathcal{C}) \subseteq \mathbb{X}$ is not restrictive for sufficiently regular linear systems with additive and multiplicative uncertainties (Blanchini & Miani, 2015, Thm. 7.2), although, in theory, it is possible to construct linear systems that merely admit an invariant but no strictly contractive set. If a strictly contractive set $\mathcal{C}$ exists, any entirely simple polytope $P(Y, \sigma)$ with

$$\sqrt{\beta} \cdot \mathcal{C} \subseteq P(Y, \sigma) \subseteq \mathcal{C} \quad (20)$$

satisfies Assumption 1 after setting $\beta \leftarrow \sqrt{\beta}$. In order to answer questions regarding the worst-case complexity of this construction in dependence on n_x and β , it is mentioned here that the problem of approximating compact convex sets with polytopes is well-studied (Chen, 2005; Schneider, 1981).

Remark 5. Let the polytope $P(Y, \sigma)$ be entirely simple but not necessarily contractive, and let $\mathcal{V}$ be its given vertex configuration. There exists a feasible β -contractive polytope $P(Y, y) \subseteq \mathbb{X}$ with $0 \leq y \in \mathbb{Y}_{\mathcal{V}}$ if and only if the convex optimization problem

$$\min_{y, u} 0 \quad \text{s.t.} \quad \begin{cases} \forall i \in \{0, \dots, \bar{m}\}, \forall j \in \{0, \dots, l\}, \\ Y \bar{A}_j V_i y + Y \bar{B}_j u_i + \bar{w} \leq \beta y \\ E y \leq 0, 0 \leq y, u_i \in \mathbb{U}, V_i y \in \mathbb{X} \end{cases} \quad (21)$$

has a feasible point (y^*, u^*) . The constraints $V_i y \in \mathbb{X}$ in (21) enforce feasibility of $P(Y, y)$, since the equivalence

$$P(Y, y) \subseteq \mathbb{X} \iff \forall i \in \{1, \dots, \bar{m}\}, V_i y \in \mathbb{X}$$

holds for all $y \in \mathbb{Y}_{\mathcal{V}}$. As such, the above statement is a direct consequence of Corollary 4. Thus, if (21) has a feasible solution (y^*, u^*) , one can simply set $\sigma \leftarrow y^*$ in order to satisfy Assumption 1—possibly after adding a small perturbation to y^* and making β slightly larger in the unlikely event that $P(Y, y^*)$ is not entirely simple. Otherwise, if (21) is infeasible, one needs to add more rows to Y and repeat the above procedure with a different σ until a feasible and strictly contractive polytope is found. Note that this procedure does not require the user to provide any robust ancillary feedback law.

Lemma 2. Let Assumption 1 hold, let $y_s \in \mathbb{R}^m$ satisfy $y_s \leq \sigma$ and $(y_s, y_s) \in \mathcal{F}$, and let

$$\gamma \stackrel{\text{def}}{=} \min_{\gamma' \geq 0} \gamma' \quad \text{s.t.} \quad \gamma'(\sigma - y_s) \geq \beta \sigma - y_s \quad (22)$$

denote a contraction constant, $\gamma < 1$. Then, there exists a sequence $y_0, y_1, \dots \in \mathbb{R}^m$ such that for all $x_0 \in P(Y, \sigma)$

- (1) the initial feasibility condition, $Yx_0 \leq y_0$, holds,
- (2) we have $(y_k, y_{k+1}) \in \mathcal{F}$ for all $k \in \mathbb{N}$, and
- (3) the sequence converges γ -exponentially; that is,

$$\forall k \in \mathbb{N}, \quad \|y_k - y_s\|_{\infty} \leq \gamma^k \|\sigma - y_s\|_{\infty}.$$

Proof. The following proof shows that the sequence

$$\forall k \in \mathbb{N}, \quad y_k \stackrel{\text{def}}{=} \gamma^k \sigma + (1 - \gamma^k) y_s \quad (23)$$

satisfies all three requirements of the lemma. We start with the case $k = 0$ for which (23) yields $y_0 = \sigma$. This choice of y_0 satisfies the first condition of the lemma, since $x_0 \in P(Y, \sigma) = P(Y, y_0)$ implies $Yx_0 \leq y_0$. Moreover, the definition of γ ensures that the inequality

$$\begin{aligned} y_{k+1} &= \gamma^{k+1} \sigma + (1 - \gamma^{k+1}) y_s = y_s + \gamma^k [\gamma(\sigma - y_s)] \\ &\stackrel{(22)}{\geq} y_s + \gamma^k [\beta \sigma - y_s] = \gamma^k \beta \sigma + (1 - \gamma^k) y_s \end{aligned} \quad (24)$$

holds. In addition, Corollary 4 and Assumption 1 imply $(\sigma, \beta \sigma) \in \mathcal{F}$ and we recall that $(y_s, y_s) \in \mathcal{F}$ holds due to the assumptions of this lemma. Consequently, we have

$$\gamma^k(\sigma, \beta \sigma) + (1 - \gamma^k)(y_s, y_s) \in \mathcal{F}, \quad (25)$$

as $\mathcal{F}$ is convex. Since $\sigma \in \mathbb{Y}_{\mathcal{V}}$ (per Assumption 1) and $y_s \in \mathbb{Y}_{\mathcal{V}}$ (as $(y_s, y_s) \in \mathcal{F}$) holds, (23) implies that

$$y_{k+1} \in \mathbb{Y}_{\mathcal{V}}, \quad (26)$$

as $\mathbb{Y}_{\mathcal{V}}$ is a convex cone. A third consequence of (23) is

$$(y_k, \gamma^k \beta \sigma + (1 - \gamma^k) y_s) \stackrel{(23)}{=} \gamma^k(\sigma, \beta \sigma) + (1 - \gamma^k)(y_s, y_s). \quad (27)$$

Thus, after substituting (27) in (25) we find that

$$(y_k, \gamma^k \beta \sigma + (1 - \gamma^k) y_s) \in \mathcal{F}$$

and then it follows from (24) and (26) that we have $(y_k, y_{k+1}) \in \mathcal{F}$. Finally, the construction of $(y_k)_{k \in \mathbb{N}}$ in (23) is such that

$$\|y_k - y_s\|_{\infty} = \|\gamma^k(\sigma - y_s)\|_{\infty} = \gamma^k \|\sigma - y_s\|_{\infty}$$

implying that the last condition of this lemma is satisfied, too. This completes our proof. $\square$

5. Polytopic tube MPC

The goal of this section is to develop a convex reformulation of (2). We specialize on configuration-constrained polytopic tubes with domain

$$\mathcal{X} = \{ P(Y, y) \mid y \in \mathbb{Y}_{\mathcal{V}} \}$$

recalling that $\mathbb{Y}_{\mathcal{V}}$ denotes the vertex configuration domain of the β -contractive polytope $P(Y, \sigma)$.

5.1. Stage cost function

Let us assume that the stage cost function has the form

$$L(P(Y, y), \mu) = \ell(y, u), \quad (28)$$

where ℓ can be any Lipschitz continuous and strictly convex function in y and u . We recall that the vertex control law μ in (17) depends on the parameter y of the current polytope and the vertex control inputs u . As such, it seems reasonable to assume that L can be expressed as in (28) for a suitable function ℓ .

Example 1. Let $\mathcal{X}$ and $V_1, V_2, \dots, V_{\bar{m}}$ be defined as before, recalling that the vertices of $P(Y, y)$ are given by $V_i y$ as long as $E y \leq 0$. The average over the vertices,

$$\bar{y} \stackrel{\text{def}}{=} \frac{1}{\bar{m}} \sum_{i=1}^{\bar{m}} V_i y = \bar{V} y \quad \text{with} \quad \bar{V} \stackrel{\text{def}}{=} \frac{1}{\bar{m}} \sum_{i=1}^{\bar{m}} V_i,$$

can be interpreted as the center of $P(Y, y)$. Similarly, if $u_1, u_2, \dots, u_{\bar{m}} \in \mathbb{U}$ denote associated vertex controls,

$$\bar{u} \stackrel{\text{def}}{=} \frac{1}{\bar{m}} \sum_{i=1}^{\bar{m}} u_i = \bar{U} u \quad \text{with} \quad \bar{U} \stackrel{\text{def}}{=} \frac{1}{\bar{m}} (\mathbb{1}, \dots, \mathbb{1})$$

² We use the definition $0^0 \stackrel{\text{def}}{=} 1$ for $\gamma = 0$.

corresponds to an average control input. An example for a practical quadratic stage cost function is then given by

$$\ell(y, u) = \|\bar{V}y\|_Q^2 + \|\bar{U}u\|_R^2 + \sum_{i=1}^{\bar{m}} \left\{ \|V_i y - \bar{V}y\|_S^2 + \|u_i - \bar{U}u\|_T^2 \right\}, \quad (29)$$

where Q, R, S , and T denote positive definite weighting matrices. Such costs model a trade-off between average tracking performance and the squared distance of the tube vertices and vertex control inputs to their average values.

5.2. Initial cost function

Formulating Tube MPC controllers which are stable in the enclosure sense requires, in general, an initial cost function L_0 , see Villanueva, De Lazzari, Müller, and Houska (2020) for details. Let us construct L_0 by introducing the 1-step cost-to-travel function (Houska & Müller, 2017)

$$V(y, y^+) \stackrel{\text{def}}{=} \min_u \ell(y, u) \quad \text{s.t.} \quad \begin{cases} \forall i \in \{1, \dots, \bar{m}\}, \forall j \in \{1, \dots, l\}, \\ Y\bar{A}_j V_i y + Y\bar{B}_j u_i + \bar{w} \leq y^+, \\ V_i y \in \mathbb{X}, u_i \in \mathbb{U}, V_i y^+ \in \mathbb{X}, \\ E y \leq 0, E y^+ \leq 0, \end{cases} \quad (30)$$

where $V(y, y^+) \stackrel{\text{def}}{=} \infty$ whenever (30) is infeasible. Next, we compute an invariant set by solving

$$V_s \stackrel{\text{def}}{=} \min_{y, y^+} V(y, y^+) \quad \text{s.t.} \quad y = y^+ \mid \lambda. \quad (31)$$

Assumption 1 ensures that (σ, σ) is a feasible point of (31). Consequently, since ℓ is strictly convex, (31) is a feasible and strictly convex optimization problem. As such, the following assumption is mild.³

Assumption 2. The strictly convex optimization problem (31) admits a unique primal solution (y_s, y_s) and a dual solution $\lambda \in \mathbb{R}^{n_x}$, such that strong duality holds.

Under Assumption 2, the rotated cost-to-travel function

$$R(y, y^+) \stackrel{\text{def}}{=} V(y, y^+) + \lambda^\top (y - y^+) - V_s \quad (32)$$

is positive definite with respect to the point (y_s, y_s) . Thus, we can define the initial cost function

$$\forall y \in \mathbb{Y}, \quad L_0(P(Y, y)) \stackrel{\text{def}}{=} \lambda^\top y.$$

This has the advantage that (2) is equivalent to

$$\min_y \sum_{k=0}^{N-1} R(y_k, y_{k+1}) + \lambda^\top y_N + L_N(P(Y, y_N)) \quad \text{s.t.} \quad \hat{x} \in P(Y, y_0), \quad (33)$$

which can be interpreted as a positive definite tracking problem in the parameter space. To achieve stability in the enclosure sense, we must ensure that the parameter sequence y_k converges to y_s —see Villanueva et al. (2020) for details.

³ If Assumption 1 holds, if ℓ is a proper, strictly convex, and radially unbounded function, and if Slater's constraint qualification is satisfied, Assumption 2 always holds (Boyd & Vandenberghe, 2004).

5.3. Terminal cost function

In this section, a stabilizing terminal cost is constructed. For this aim, we assume that the initial polytope $P(Y, \sigma)$ is large enough to contain the optimal invariant set,

$$y_s \leq \sigma. \quad (34)$$

Moreover, we solve (30) offline at $(y, y^+) = (\sigma, \sigma)$ and at $(y, y^+) = (y_s, y_s)$. The corresponding minimizers are denoted by u_σ and u_s , such that

$$\ell(\sigma, u_\sigma) = V(\sigma, \sigma) = R(\sigma, \sigma) + V_s \quad (35)$$

$$\text{and} \quad \ell(y_s, u_s) = V(y_s, y_s) = V_s. \quad (36)$$

Let $\bar{\ell}$ denote the Lipschitz constant of ℓ on $P(Y, \sigma)$. In the following, we introduce the shorthand

$$\rho \stackrel{\text{def}}{=} \bar{\ell} \left\| \begin{pmatrix} \sigma - y_s \\ u_\sigma - u_s \end{pmatrix} \right\|_2 + (1 - \gamma) |\lambda^\top (\sigma - y_s)|, \quad (37)$$

which can be pre-computed offline. Here, γ is computed as in Lemma 2, see Eq. (22). Next, our goal is to show that the cost function

$$L_N(P(Y, y)) \stackrel{\text{def}}{=} -\lambda^\top y + M(y)$$

with auxiliary function

$$M(y) \stackrel{\text{def}}{=} \min_{\alpha \in [0, 1]} R(y, y_s + \alpha(\sigma - y_s)) + \frac{\rho \cdot \alpha}{1 - \gamma} \quad (38)$$

can be used as stabilizing terminal cost. Note that M is convex, since R is convex in both arguments. The following theorem establishes that M is a Lyapunov function.

Theorem 3. Let Assumption 1 and 2 be satisfied and let M be defined as in (38). The function M is non-negative and satisfies $M(y) = 0$ if and only if $y = y_s$. Moreover, the Lyapunov descent condition holds,

$$\forall y \in \mathbb{R}^m, \quad \min_{y^+} \{R(y, y^+) + M(y^+)\} \leq M(y).$$

Proof. Note that (38) implies $M(y) \geq 0$ for all $y \in \mathbb{R}^m$, since R is positive definite and $\frac{\rho \cdot \alpha}{1 - \gamma}$ is non-negative. Moreover, $M(y_s) = 0$. This follows from (38) by substituting $\alpha = 0$ and using that $R(y_s, y_s) = 0$. Our next goal is to show that $M(y) > 0$ for all $y \neq y_s$. In order to see this, assume that $M(y) = 0$. In this case, we must have $\alpha = 0$, because, otherwise, the second term in (38) would be strictly positive. Consequently, we must have $R(y, y_s) = 0$, but this is impossible for $y \neq y_s$, since y_s is—due to Assumption 2—the unique minimizer of $\min_y R(y, y_s)$. Thus, M is a positive definite function.

Next, in analogy to the construction in the proof of Lemma 2, for an arbitrary parameter $\alpha \in [0, 1]$, we introduce the auxiliary sequences

$$\tilde{y}_k \stackrel{\text{def}}{=} y_s + \alpha \gamma^k (\sigma - y_s) \quad (39)$$

$$\text{and} \quad \tilde{u}_k \stackrel{\text{def}}{=} u_s + \alpha \gamma^k (u_\sigma - u_s), \quad (40)$$

such that $(\tilde{y}_k, \tilde{y}_{k+1}) \in \mathcal{F}$. Since $\tilde{u}_k$ is a feasible point of (30) at $(y, y^+) = (\tilde{y}_k, \tilde{y}_{k+1})$, we find that

$$V(\tilde{y}_k, \tilde{y}_{k+1}) \leq \ell(\tilde{y}_k, \tilde{u}_k). \quad (41)$$

This inequality leads us to the upper bound

$$\begin{aligned} R(\tilde{y}_0, \tilde{y}_1) &\stackrel{(32)}{=} V(\tilde{y}_0, \tilde{y}_1) + \lambda^\top (\tilde{y}_0 - \tilde{y}_1) - V_s \\ &\stackrel{(41)}{\leq} \ell(\tilde{y}_0, \tilde{u}_0) + \lambda^\top (\tilde{y}_0 - \tilde{y}_1) - \ell(y_s, u_s) \\ &\leq \bar{\ell} \cdot \left\| \begin{pmatrix} \tilde{y}_0 - y_s \\ \tilde{u}_0 - u_s \end{pmatrix} \right\|_2 + |\lambda^\top (\tilde{y}_0 - \tilde{y}_1)| \end{aligned}$$

$$\begin{aligned}
&\stackrel{(39), (40)}{\leq} \alpha \cdot \bar{\ell} \cdot \left\| \begin{pmatrix} \sigma - y_s \\ u_\sigma - u_s \end{pmatrix} \right\|_2 \\
&\quad + \alpha \cdot |\lambda^\top (\sigma - y_s) - \gamma \lambda^\top (\sigma - y_s)| \\
&\stackrel{(37)}{=} \alpha \cdot \rho,
\end{aligned} \tag{42}$$

which, in turn, implies that the inequality

$$\begin{aligned}
M(\tilde{y}_0) &\stackrel{(38)}{=} \min_{\alpha' \in [0,1]} \left\{ R(\tilde{y}_0, y_s + \alpha'(\sigma - y_s)) + \frac{\rho \cdot \alpha'}{1 - \gamma} \right\} \\
&\leq R(\tilde{y}_0, y_s + \alpha \gamma (\sigma - y_s)) + \frac{\rho \cdot \alpha \cdot \gamma}{1 - \gamma} \\
&\stackrel{(39)}{=} R(\tilde{y}_0, \tilde{y}_1) + \frac{\rho \cdot \alpha \cdot \gamma}{1 - \gamma} \stackrel{(42)}{\leq} \frac{\rho \cdot \alpha}{1 - \gamma}
\end{aligned} \tag{43}$$

holds for any choice of $\alpha \in [0, 1]$ (keeping in mind that $\tilde{y}_0$ depends on that value of α , too). Consequently, if we choose α as in (38) we find that

$$\begin{aligned}
M(y) &\stackrel{(38)}{=} R(y, \tilde{y}_0) + \frac{\rho \cdot \alpha}{1 - \gamma} \\
&\stackrel{(43)}{\geq} R(y, \tilde{y}_0) + M(\tilde{y}_0) \geq \min_{y^+} \{ R(y, y^+) + M(y^+) \}.
\end{aligned} \tag{44}$$

The latter inequality corresponds to the Lyapunov descent statement of the theorem. $\square$

Remark 6. Note that existing attempts to construct terminal cost functions for HTMPC (Raković, Kouvaritakis, Findeisen, & Cannon, 2012) or ETMPC (Raković et al., 2016) rely on the availability of robustly stabilizing affine feedback laws. This is in contrast to the above construction of M , for which it is not required that a robustly stabilizing affine feedback law exists.

Remark 7. The Lyapunov function M is generic as no assumptions on ℓ apart from strict convexity and Lipschitz continuity are needed. However, the bound of R in (42) is not sharp. As such, M is not necessarily the best possible approximation of the infinite horizon control performance, which could be found by using sharper cost estimates. For instance, if ℓ is a quadratic form, it is not difficult to derive bounds that are tighter than (42).

5.4. Implementation of CCTMPC

Since $\mathbb{X}$, $\mathbb{U}$, ℓ and M are convex, (2) is equivalent to the convex optimization problem

$$\begin{aligned}
\min_{y, u} \quad & \lambda^\top y_0 + \sum_{k=0}^{N-1} \ell(y_k, u_k) - \lambda^\top y_N + M(y_N) \\
\text{s.t.} \quad & \begin{cases} \forall k \in \{0, \dots, N-1\}, \\ \forall i \in \{1, \dots, \bar{m}\}, \forall j \in \{1, \dots, l\}, \\ Y \bar{A}_j V_i y_k + Y \bar{B}_j u_{k,i} + \bar{w} \leq y_{k+1}, \\ E y_k \leq 0, u_{k,i} \in \mathbb{U}, V_i y_k \in \mathbb{X}, Y \hat{x} \leq y_0. \end{cases}
\end{aligned} \tag{45}$$

Note that the optimization variables of (45) are the tube parameters $y = [y_0, y_1, \dots, y_N]$ and the vertex control inputs $u = [u_0, u_1, \dots, u_{N-1}]$.

Remark 8. Assuming that $\mathbb{U}$ and $\mathbb{X}$ are polyhedra with n_U and n_X facets respectively, Problem (45) has

$$(N+1) \cdot m + N \cdot \bar{m} \cdot n_u$$

optimization variables as well as

$$m \cdot \bar{m} \cdot l \cdot N + m_E \cdot N + \bar{m} \cdot (n_X + n_U) \cdot N + m$$

inequality constraints,⁴ where $m_E \ll m \cdot \bar{m}$ denotes the number of non-redundant rows of E ; see Remark 2 for details. As such, under the mentioned assumptions, one may state that (45) has polynomial run-time complexity. However, it should also be clear that solving (45) is computationally demanding, since the product of m , $\bar{m}$, l , and N is potentially a very large number.

5.5. Stability analysis

Let $y^*(\hat{x})$ and $u^*(\hat{x})$ be parametric minimizers of (45) in dependence on the measurement $\hat{x}$. Next, due to the construction of y^* and u^* , any control law of the form

$$\mu_{\text{MPC}}(\hat{x}) \stackrel{\text{def}}{=} \sum_{i=1}^{\bar{m}} \theta_i(\hat{x}) u_{0,i}^*(\hat{x}) \tag{46}$$

can be used as a feasible MPC feedback control law, as long as the scalar coefficient functions θ_i satisfy

$$\hat{x} = \sum_{i=1}^{\bar{m}} \theta_i(\hat{x}) V_i y_0^*(\hat{x}), \quad \theta(\hat{x}) \geq 0 \quad \text{and} \quad \|\theta(\hat{x})\|_1 = 1,$$

recalling the construction of μ in the proof of Corollary 4, see Eq. (17). For instance, one can pick

$$\begin{aligned}
\theta(\hat{x}) &\in \underset{\theta}{\operatorname{argmin}} \left\| \sum_{i=1}^{\bar{m}} \theta_i u_{0,i}^*(\hat{x}) \right\|_2^2 \\
\text{s.t.} \quad & \begin{cases} \hat{x} = \sum_{i=1}^{\bar{m}} \theta_i V_i y_0^*(\hat{x}), \\ \theta \geq 0, \sum_{i=1}^{\bar{m}} \theta_i = 1, \end{cases}
\end{aligned} \tag{47}$$

but using other convex control penalty functions is, of course, possible, too. The associated uncertain closed-loop system of (45) has the form

$$\forall k \in \mathbb{N}, \quad x_{k+1} = A_k x_k + B_k \mu_{\text{MPC}}(x_k) + C w_k,$$

which is started at a given initial state $x_0 \in \mathbb{R}^{n_x}$. The following theorem establishes recursive feasibility and asymptotic stability, in the enclosure sense, of the controller—see Villanueva et al. (2020) for a discussion of set-theoretic stability in Tube MPC schemes.

Theorem 4. Let Assumption 1 and 2 be satisfied, let (y_s, λ) be given by solving (31), and let M be given by (38). Then the following statements hold independently of the sequences $w_0, w_1, \dots \in \mathbb{W}$ and $[A_0, B_0], [A_1, B_1], \dots \in \Delta$.

- (1) If (45) is feasible for the initial state, $\hat{x} = x_0$, then it is also feasible for all future measurements, $\hat{x} = x_k$.
- (2) We have $x_k \in P(Y, y_0^*(x_k))$ for all $k \in \mathbb{N}$.
- (3) The sequence $y^*(x_k)$ is asymptotically stable,

$$\lim_{k \rightarrow \infty} y^*(x_k) = [y_s, y_s, \dots, y_s].$$

Proof. Let us introduce the non-negative function

$$\mathcal{L}(y) \stackrel{\text{def}}{=} \sum_{i=0}^{N-1} R(y_i, y_{i+1}) + M(y_N),$$

where R and M are defined as in (32) and (38). Moreover, we have $\mathcal{L}(y) = 0$ if and only if $y = y_s$, which follows from Theorem 3 and the positive definiteness of R . Now, let us assume that (45)

⁴ In order to be precise, it should be mentioned that, depending on how the function M is represented, additional auxiliary optimization variables and constraints are needed to arrive at a practical implementation; see (30) and (38).

is feasible at $\hat{x} = x_k$ such that $y^*(x_k)$ is well-defined. In this case, we can define

$$\tilde{y} \stackrel{\text{def}}{=} [y_1^*(x_k), \dots, y_N^*(x_k), y_N^+], \quad (48)$$

where the last element y_N^+ of $\tilde{y}$ is chosen such that

$$R(y_N^*(x_k), y_N^+) + M(y_N^+) \leq M(y_N^*(x_k)). \quad (49)$$

The existence of such a vector y_N^+ is ensured by [Theorem 3](#). Moreover, a feasible shifted sequence for the vertex control variables of (45) is given by

$$\tilde{u} \stackrel{\text{def}}{=} [u_1^*(x_k), \dots, u_{N-1}^*(x_k), \hat{u}^*], \quad (50)$$

where $\hat{u}^*$ is the optimal vertex control inputs that needs to be computed when evaluating $R(y_N^*(x_k), y_N^+)$ (see (30) and (32)). This construction is such that $(\tilde{y}, \tilde{u})$ is feasible for (45) at the next time instance, as $\hat{x} = x_{k+1}$ satisfies

$$x_{k+1} \in P(Y, y_1^*(x_k)),$$

independently of w_k (due to the above definition of μ_{MPC}), and hence $Y\hat{x} \leq \tilde{y}_0 = y_1^*(x_k)$. That is, $\hat{x}$ is contained in the robust forward invariant tube. Thus, Statement (1) holds independent of the uncertainty realization, and it follows that Statement (2) holds. Moreover,

$$\mathcal{L}(y^*(x_k)) \stackrel{(49)}{\geq} r_k + \mathcal{L}(\tilde{y}) \geq r_k + \mathcal{L}(y^*(x_{k+1})), \quad (51)$$

where we have introduced the shorthand

$$r_k \stackrel{\text{def}}{=} R(y_0^*(x_k), y_1^*(x_k)) \geq 0.$$

The last inequality in (51) follows from the optimality of $y^*(x_{k+1})$, since our definition of $\mathcal{L}$ is such that this function coincides with the optimal value function of our MPC controller. Because r_k is non-negative and because we have $r_k = 0$ if and only if $y_0^*(x_k) = y_s$, the function $\mathcal{L}$ in (51) is a strictly descending Lyapunov function. This implies that Statement (3) holds. $\square$

6. Tutorial examples and numerical illustration

This section compares CCTMPC with RTMPC, HTMPC, ETMPC, DAFMPC, and FTMPC for sufficiently regular linear systems that admit a strictly contractive polytope, such that [Assumption 1](#) and [2](#) hold.

- CCTMPC is never more conservative than polytopic RTMPC: one can always use the invariant cross-section of the rigid tube to construct a vertex configuration domain for CCTMPC. This implies that CCTMPC admits a more flexible tube representation.
- Using the same argument, CCTMPC is never more conservative than HTMPC: the vertex configuration constraint is invariant under homothetic scaling, since $\mathbb{Y}_V$ is a convex cone. This statement is independent of which control law parameterization is used for the implementation of HTMPC. It implies in particular that CCTMPC is never more conservative than the vertex control input optimization based variants of polytopic MPC that are proposed in [Langson et al. \(2004, Section 3.3\)](#) or the variant based on piecewise affine feedback parameterization proposed in [Raković et al. \(2022, Section 6 B\)](#).
- For $n_x = 2$, CCTMPC is never more conservative than ETMPC as long as the vertex configuration domain is constructed as in [Remark 3](#). In general, for $n_x \geq 3$, however, the constraint $Ey \leq 0$ leads to an actual restriction on the class of representable polytopes. This is in contrast to ETMPC, which does not introduce such a configuration constraint. On the other hand, the construction of ETMPC controllers relies on

an affine parameterization of the feedback control policy, while CCTMPC introduces no such restriction; see [Remark 4](#). Moreover, existing set-propagation approaches for ETMPC are conservative, complicating the comparison even further. Nevertheless, numerical comparisons indicate that CCTMPC is less conservative than ETMPC—at least for the benchmark examples in $\mathbb{R}^3$ and $\mathbb{R}^4$ that we will propose below.

- It is possible to construct systems with four states for which CCTMPC is systematically less conservative than any separable state feedback controller such as DAFMPC and FTMPC; see [Section 6.3](#).

In addition to the above comments about conservatism, it should be pointed out that RTMPC, HTMPC, ETMPC, and CCTMPC have in common that their computational complexity is $\mathcal{O}(N)$, where N is the prediction horizon. In contrast to this, the computational complexity of DAFMPC and FTMPC is $\mathcal{O}(N^2)$, due to an entirely different parameterization strategy.

6.1. Example 1

Our first example considers a system with

$$A = \frac{1}{5} \begin{pmatrix} 5 & 1 \\ -1 & 4 \end{pmatrix}, \quad B = \frac{1}{5} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad C = \frac{1}{5} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The four vertices of Δ are given by

$$[\bar{A}_j, \bar{B}_j] = \left[A \pm \frac{1}{20} e_1 e_1^T, B \pm \frac{1}{50} e_2 \right]$$

with e_1, e_2 being the standard basis vectors of $\mathbb{R}^2$. The state, control and uncertainty bounds are given by

$$\mathbb{X} = [-10, 7.7] \times [-5.7, 10], \quad \mathbb{U} = [-10, 10],$$

$$\text{and} \quad \mathbb{W} = [-0.5, 0.5] \times [-2, 2].$$

We start with a regular polytope with $m \geq 3$ facets and choose Y and σ as in [Remark 3](#) with

$$\varphi_1 = 0, \quad \varphi_2 = \frac{2\pi}{m}, \quad \varphi_3 = \frac{4\pi}{m}, \dots, \quad \varphi_m = \frac{2(m-1)\pi}{m}.$$

Here, $P(Y, \sigma)$ is regular and entirely simple, but it is neither contractive nor invariant. However, by [Remark 5](#), specifically, by solving (21) for $\beta = 0.975$, β -contractive polytopes can be found for any $m \geq 7$. An optimal vertex configuration domain can be found by computing E as explained in [Remark 3](#).

Let us choose ℓ as in (29) with $Q = S = \mathbf{1}$, $R = \mathbf{1}$, and $T = 0.05$. Additionally, the prediction horizon is set to $N = 50$. The left part of [Fig. 2](#) shows the predicted tube obtained by solving (45) for the state measurement

$$x_0 = \hat{x} = [4, 8]^T$$

and $m = \bar{m} = 18$. The shape and size of the tube cross sections change in time as all tube parameters are optimized. This illustrates the advantage of CCTMPC compared to RTMPC. The tube converges to the optimal invariant polytope $P(Y, y_s)$, shown in blue. The optimal vertex control inputs u_s are shown as red dots in the middle part of [Fig. 2](#). The black dotted lines show their relation to the vertices of $P(Y, y_s)$ —shown in blue. Note that it is impossible to interpolate all vertex control inputs with one hyperplane. The gray shaded areas correspond, however, to one possible continuous piecewise affine control law interpolating all vertex inputs as well as the central input, $\bar{u} = 0$ (visualized as green dot).

The statement of [Theorem 4](#) can be verified numerically by running the above MPC control setup with $m = 18$ in a closed loop for random uncertainty scenarios. The right part of [Fig. 2](#) shows an evaluation of the Lyapunov function $\mathcal{L}(y^*(x_k))$ along the

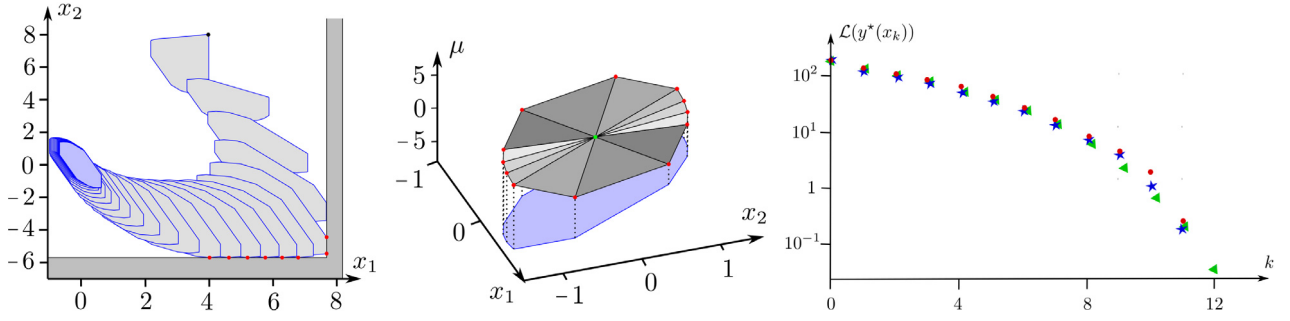


Fig. 2. LEFT: Predicted tube (light gray shaded polytopes) for $\hat{x} = [4, 8]^T$ (black dot) and $m = 18$. The prediction horizon has been set to $N = 50$. The optimal invariant set $P(Y, y_s)$ is indicated by the dark blue shaded set. It is a simple polytope, but it is not an entirely simple polytope, because it has $12 < 18$ vertices. MIDDLE: The optimal vertex control inputs (red dots) of the optimal invariant polytope (blue shaded set). RIGHT: Lyapunov function $\mathcal{L}$ along the closed-loop trajectory x_k for three random uncertainty scenarios w . In the first scenario (red dots) and the third scenario (blue stars), the closed-loop state reaches the optimal invariant set after 12 iterations; that is, $\mathcal{L}(y^*(x_k)) = 0$ for all $k \geq 12$ (not visualized, as we use a logarithmic scale). For the second scenario, the invariant set is reached after 13 iterations. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

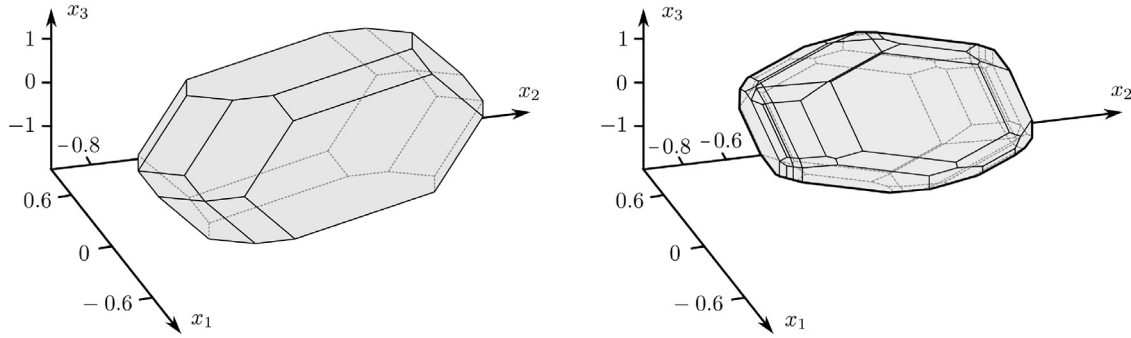


Fig. 3. Optimal invariant polytopes for the system from Example 2. The left polytope has been found for a matrix Y with $m = 26$ rows and an associated vertex configuration domain with up to $\bar{m} = 48$ vertices, while the right polytope is obtained for a setting with $m = 124$ and $\bar{m} = 336$. Its volume is approximately equal to one third of the volume of the left polytope.

MPC closed-loop trajectories x_k for three such randomly chosen scenarios. The specific value of $\mathcal{L}$ —and, thus, the actual control performance—depends on the uncertainty. However, independently of how this uncertainty is chosen, $\mathcal{L}$ is always strictly monotonically descending until it reaches 0, as confirmed in all our tests.

6.2. Example 2

This section discusses an example in $\mathbb{R}^3$, given by

$$A = \frac{1}{2} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad B = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad C = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

again with $[A, B] = [\bar{A}_1, \bar{B}_1]$ being given, as well as

$$\mathbb{X} = [-5, 5]^3, \quad \mathbb{U} = [-10, 10], \quad \text{and} \quad \mathbb{W} = [-0.2, 0.2]^3.$$

We choose ℓ as in (29) with $Q = \mathbb{I}$, $S = 7 \cdot \mathbb{I}$, $R = 1$, and $T = 0.01$. The matrix Y has row vectors of the form

$$\frac{[i, j, k]}{\sqrt{i^2 + j^2 + k^2}}, \quad (52)$$

where $(i, j, k) \in \{-1, 0, 1\}^3 \setminus \{(0, 0, 0)\}$. We initially set $\sigma = [1, 1, \dots, 1]^T$ in order to compute a vertex configuration domain and a contractive polytope as explained in Remark 5. This leads to a contractive polytope with

$$m = 3^3 - 1 = 26$$

facets and $\bar{m} = 48$ vertices. Its reduced configuration matrix E has 48 rows and 168 non-zero coefficients. The left part of Fig. 3

shows a 3-dimensional visualization of the corresponding optimal invariant set. To illustrate how this result can be improved by increasing the number of facet directions and vertices, the right part of Fig. 3 shows a less conservative optimal invariant set constructed with $m = 5^3 - 1 = 124$ facet directions and $\bar{m} = 336$ vertices. Its reduced configuration matrix E has 290 rows and 1031 non-zero coefficients. Here, the rows of Y have also been constructed as in (52) but with $(i, j, k) \in \{-2, -1, 0, 1, 2\}^3 \setminus \{(0, 0, 0, 0, 0)\}$.

Numerical experiments indicate that ETMPC can be used to generate similar polytopes for both choices of Y . These are, however, at least 1% sub-optimal for the best possible affine control law (in terms of stage cost value) that we managed to compute by exhaustive search. Without optimizing the affine feedback law, ETMPC is usually more than 1% sub-optimal.

6.3. Example 3

Our last example shows that CCTMPC can outperform both DAFMPC and FPMPC in terms of conservatism and run-time complexity. We consider the control system

$$\begin{aligned} x_1^+ &= \frac{1}{2}x_3 - \frac{1}{2}x_4 \\ x_2^+ &= \frac{1}{2}x_3 + \frac{1}{2}x_4 + u \\ x_3^+ &= x_4 \\ x_4^+ &= w \end{aligned} \quad \text{with} \quad \begin{cases} \mathbb{X} = \mathbb{R}^4 \\ \mathbb{U} = [0, 1] \\ \mathbb{W} = [-1, 1] \end{cases} \quad (53)$$

Here, $x \in \mathbb{R}^4$ denotes the state at a given time and x^+ the corresponding successor state. Sub-indices denote the components of the states (not a time index). Our goal is to minimize the least-squares distance of all states to 0, for example, by setting

$\ell(y, u) = y^2$, but other choices of ℓ are possible, too. It is not difficult to see that an optimal feedback law is in this case given by

$$\mu(x) = \begin{cases} 0 & \text{if } x_3 + x_4 > 0 \\ -\frac{x_3+x_4}{2} & \text{if } x_3 + x_4 \in [-2, 0] \\ 1 & \text{otherwise,} \end{cases} \quad (54)$$

because only x_2 depends on u . Next, we set

$$Y = \begin{pmatrix} 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & -1 & 2 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}^T \in \mathbb{R}^{6 \times 4}$$

and $\sigma = (1, 1, 1, 1, 1, 1)^T$. We use the procedure from Remark 5 to compute a contractive polytope and its associated reduced vertex configuration template,⁵

$$E = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -1 & -1 & -2 \end{pmatrix}.$$

If we minimize $\ell(y, u) = y^2$, the optimal invariant polytope $P(Y, y_s)$ is given by $y_s = (1, 1, 0, 1, 1, 0)^T$. Note that $P(Y, y_s)$ is a four dimensional polytope with 6 facets and 8 vertices. Its vertex control inputs $u_s \in \mathbb{R}^8$ are unique and given by $(u_s)_i = \mu(V_i y_s)$ for $i \in \{1, 2, \dots, 8\}$, where μ denotes the optimal control law in (54). In other words, the proposed CCTMPC method is able to find the optimal nonlinear control law. As such, for this example and this choice of Y one may state that CCTMPC finds the best possible invariant polytope.

Our next goal is to analyze an FPTMPC controller for (53). Note that the matrix A of (53) is nil-potent, $A^3 = 0$. Thus, we may set the prediction horizon of the controller to $N = 3$ (without loss of generality), as the system does anyhow neither remember uncertainties nor control inputs for longer than 3 time steps. Next, let us work out the fully parameterized extreme partial state trajectories. They are given by

$$\begin{aligned} \tilde{x}^{0,3} &= (0, \tilde{u}^{0,2}, 0, 0)^T, \\ \tilde{x}^{1,1,3} &= (0.5, 0.5 + \tilde{u}^{1,1,2}, 0, 0)^T, \\ \tilde{x}^{2,1,3} &= (-0.5, -0.5 + \tilde{u}^{2,1,2}, 0, 0)^T, \\ \tilde{x}^{1,2,3} &= (-0.5, 0.5 + \tilde{u}^{1,2,2}, 1, 0)^T, \\ \tilde{x}^{2,2,3} &= (0.5, -0.5 + \tilde{u}^{2,2,2}, -1, 0)^T, \\ \tilde{x}^{1,3,3} &= (0, 0, 0, 1)^T, \\ \text{and } \tilde{x}^{2,3,3} &= (0, 0, 0, -1)^T, \end{aligned} \quad (55)$$

where we use the same index convention as in Raković, Kouvaritakis, Cannon, et al. (2012). That is, $\tilde{x}^{i,k,3}$ is the extreme partial state trajectory at time 3, obtained by exciting the system at time $k - 1$ with the extreme input $\tilde{w}_i = (-1)^{i-1}$. The value of the measurement $\tilde{x}^{0,0} = \hat{x}$ at time 0 is irrelevant, as the state at time 3 is independent of $\hat{x}$. The corresponding inputs,

$$\tilde{u} = (\tilde{u}^{0,2}, \tilde{u}^{1,1,2}, \tilde{u}^{2,1,2}, \tilde{u}^{1,2,2}, \tilde{u}^{2,2,2})^T \in \mathbb{R}^5,$$

need to satisfy the control constraints for all scenarios,

$$\tilde{u}^{0,2} + \tilde{u}^{1,1,2} + \tilde{u}^{1,2,2} \in [0, 1] \quad (56)$$

$$\tilde{u}^{0,2} + \tilde{u}^{1,1,2} + \tilde{u}^{2,2,2} \in [0, 1] \quad (57)$$

$$\tilde{u}^{0,2} + \tilde{u}^{2,1,2} + \tilde{u}^{1,2,2} \in [0, 1] \quad (58)$$

$$\text{and } \tilde{u}^{0,2} + \tilde{u}^{2,1,2} + \tilde{u}^{2,2,2} \in [0, 1]. \quad (59)$$

⁵ For this particular example, the vertex configuration constraint is not restrictive: there exists for every $y' \in \mathbb{R}^6$ a parameter $y \in \mathbb{R}^6$ with $Ey \leq 0$ such that $P(Y, y') = P(Y, y)$.

Recall that $P(Y, y_s)$ denotes the optimal invariant set, which has 6 facets and 8 vertices, as discussed above. Let us assume that FPTMPC was not conservative. In this case, it would be possible to find a parameter $\tilde{u}$ satisfying (56)–(59) and ensuring that all possible extreme state trajectories are contained in $P(Y, y_s)$; that is,

$$\tilde{x}^{0,3} + \tilde{x}^{1,1,3} + \tilde{x}^{1,2,3} + \tilde{x}^{i,3,3} \in P(Y, y_s), \quad (60)$$

$$\tilde{x}^{0,3} + \tilde{x}^{1,1,3} + \tilde{x}^{2,2,3} + \tilde{x}^{i,3,3} \in P(Y, y_s), \quad (61)$$

$$\tilde{x}^{0,3} + \tilde{x}^{2,1,3} + \tilde{x}^{1,2,3} + \tilde{x}^{i,3,3} \in P(Y, y_s), \quad (62)$$

$$\tilde{x}^{0,3} + \tilde{x}^{2,1,3} + \tilde{x}^{2,2,3} + \tilde{x}^{i,3,3} \in P(Y, y_s) \quad (63)$$

for all $i \in \{1, 2\}$. Conditions (56) and (60) necessarily imply Condition (64), as stated below. Analogous necessary conditions of feasibility are given by

$$\tilde{u}^{0,2} + \tilde{u}^{1,1,2} + \tilde{u}^{1,2,2} \stackrel{(56), (60)}{=} 0, \quad (64)$$

$$\tilde{u}^{0,2} + \tilde{u}^{1,1,2} + \tilde{u}^{2,2,2} \stackrel{(57), (61)}{=} 0, \quad (65)$$

$$\tilde{u}^{0,2} + \tilde{u}^{2,1,2} + \tilde{u}^{1,2,2} \stackrel{(58), (62)}{=} 0, \quad (66)$$

$$\text{and } \tilde{u}^{0,2} + \tilde{u}^{2,1,2} + \tilde{u}^{2,2,2} \stackrel{(59), (63)}{=} 1. \quad (67)$$

Subtracting (65) from (64) yields $\tilde{u}^{1,2,2} = \tilde{u}^{2,2,2}$. By substituting this equation in (66) and subtracting it from (67) we find that $1 = 0$. Clearly, this is a contradiction, which implies that FPTMPC is conservative and this result is independent of how one chooses the prediction horizon $N \geq 3$ and the initial measurement $\hat{x}$.

In order to explain why CCTMPC is for the current example less conservative than DAFMPC, it is helpful to note that the optimal feedback law (54) is equivalent to a nonlinear disturbance feedback control law,

$$\tilde{\mu}(w^-, w^{--}) = \begin{cases} 0 & \text{if } w^- + w^{--} > 0 \\ -\frac{w^- + w^{--}}{2} & \text{if } w^- + w^{--} \in [-2, 0] \\ 1 & \text{otherwise,} \end{cases}$$

where w^- denotes the last and w^{--} the disturbance from two time steps ago. By evaluating this feedback law at the extreme scenarios, $\tilde{\mu}(-1, 1) = 0$, $\tilde{\mu}(1, 1) = 0$, $\tilde{\mu}(-1, -1) = 1$, $\tilde{\mu}(1, -1) = 0$, one finds that it is impossible to interpolate these 4 uniquely optimal function values with a single affine function. This observation implies that, for this example, CCTMPC is strictly less conservative than DAFMPC.

7. Conclusions

This paper has presented a novel class of Tube MPC controllers for linear systems with additive and multiplicative uncertainty. The corresponding technical developments built upon Theorem 2, which features a variant of the Gauss–Bonnet theorem in order to simultaneously parameterize the facets and vertices of configuration-constrained polytopes. The relevance of this geometrical construction is that it enables us to freely optimize configuration-constrained robust forward invariant tubes and their associated vertex control laws via the convex optimization problem (45). Conditions under which CCTMPC is asymptotically stable and recursively feasible are established in Theorems 3 and 4.

CCTMPC is never more conservative than RTMPC, HTMPC, and, in many cases, also ETMPC. We have constructed examples for which CCTMPC is less conservative than FTPMPC and DAFMPC. For sufficiently long time horizons, CCTMPC is computationally less demanding than FTPMPC and DAFMPC.

Finally, as much as this paper has made a first step towards exploiting the face configurations of selected classes of polytopes for the development of robust control methods, a more in-depth

discussion of how to design and select suitable polytopic configurations in higher dimensional spaces has been beyond its scope. The development of algorithms and software for a systematic construction of such configuration templates and a discussion of more general questions about the complexity of robust control of constrained linear systems could be a relevant topic for future research.

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Appendix A. Proof of Theorem 1 and Corollary 3

We first show that the second statement of [Theorem 1](#) implies the first statement. The reverse implication is established in Part II, which will also imply the statement of [Corollary 3](#).

Part I. Let $\sigma \in \mathbb{Y}_{\mathcal{I}}$ be such that $P(Y, \sigma)$ is an entirely simple polyhedron and let $I \in \mathcal{I}$ be a given index set. First, we show

$$J = \left\{ j \notin I \mid 0 = \min_{x \in \mathcal{F}_I(\sigma)} Y_j x - \sigma_j \right\} = \emptyset.$$

Note that $\mathcal{F}_{I \cup J}(\sigma) = \mathcal{F}_I(\sigma)$ holds by construction and both sets are non-empty, since $\sigma \in \mathbb{Y}_{\mathcal{I}}$ and $I \in \mathcal{I}$. Moreover, there exists a point $x^* \in \mathcal{F}_{I \cup J}(\sigma)$ that satisfies

$$\forall k \notin J \cup I, \quad Y_k x^* - y_k < 0, \quad (\text{A.1})$$

which directly follows from the definition of J . Next, let

$$\mathcal{T}_{I \cup J}(\sigma) = \left\{ x \in \mathbb{R}^n \mid Y_{I \cup J} x = \sigma_{I \cup J} \right\} \quad (\text{A.2})$$

$$\text{and} \quad \mathcal{T}_I(\sigma) = \left\{ x \in \mathbb{R}^n \mid Y_I x = \sigma_I \right\} \quad (\text{A.3})$$

denote the linear subspaces of $\mathbb{R}^n$ corresponding to $I \cup J$ and I . Since $P(Y, \sigma)$ is entirely simple, [\(A.1\)](#) implies that $\mathcal{F}_{I \cup J}(\sigma)$ has non-empty interior in $\mathcal{T}_{I \cup J}(\sigma)$, implying that the dimension of the subspace $\mathcal{T}_{I \cup J}(\sigma)$ and the face $\mathcal{F}_{I \cup J}(\sigma)$ must coincide, $\dim(\mathcal{F}_{I \cup J}(\sigma)) = \dim(\mathcal{T}_{I \cup J}(\sigma))$. Consequently, since $\mathcal{F}_{I \cup J}(\sigma) = \mathcal{F}_I(\sigma)$, we have

$$\begin{aligned} n - \text{rank}(Y_I) &= \dim(\mathcal{T}_I(\sigma)) \geq \dim(\mathcal{F}_I(\sigma)) = \dim(\mathcal{F}_{I \cup J}(\sigma)) \\ &= \dim(\mathcal{T}_{I \cup J}(\sigma)) = n - \text{rank}(Y_{I \cup J}). \end{aligned} \quad (\text{A.4})$$

Since $P(Y, \sigma)$ is entirely simple, the faces $\mathcal{F}_I$ and $\mathcal{F}_{I \cup J}$ satisfy

$$\text{rank}(Y_I) = |I| \quad (\text{A.5})$$

$$\text{and} \quad \text{rank}(Y_{I \cup J}) = |I \cup J| = |I| + |J|, \quad (\text{A.6})$$

where [\(A.6\)](#) holds, as I and J are, by construction, disjoint. Now, we can substitute [\(A.5\)](#) and [\(A.6\)](#) in [\(A.4\)](#), which yields

$$n - |I| \geq n - |I| - |J| \implies |J| = 0.$$

Consequently, $J = \emptyset$. In other words, there exists a point $x^* \in \mathcal{F}_I(\sigma)$ such that

$$\forall j \notin I, \quad Y_j x^* < \sigma_j \quad \text{and} \quad Y_I x^* = \sigma_I. \quad (\text{A.7})$$

Next, let us perturb σ by a small vector δ . Since Y_I has full rank, $x_\delta^* \stackrel{\text{def}}{=} x^* + Y_I^\dagger \delta$ is well-defined for any perturbation $\delta \in \mathbb{R}^m$. Here,

$Y_I^\dagger$ denotes the right pseudo-inverse of Y_I . Since [\(A.7\)](#) holds, there exists a small $\epsilon_I > 0$ such that

$$\forall j \notin I, \quad Y_j x_\delta^* < \sigma_j + \delta_j \quad \text{and} \quad Y_I x_\delta^* = \sigma_I + \delta_I. \quad (\text{A.8})$$

for all $\delta \in \mathbb{R}^m$ with $\|\delta\| < \epsilon_I$. Consequently, the face $\mathcal{F}_I(\sigma + \delta)$ is non-empty for any such small perturbation. Thus, if we set $0 < \epsilon \stackrel{\text{def}}{=} \min_{I \in \mathcal{I}} \epsilon_I$, it follows that $\sigma + \delta \in \mathbb{Y}_{\mathcal{I}}$ for all $\delta \in \mathbb{R}^m$ with $\|\delta\| \leq \epsilon$. Hence, $\mathbb{Y}_{\mathcal{I}}$ has a non-empty interior in $\mathbb{R}^m$. It is a regular configuration domain, concluding the first part of the proof.

Part II. Let us introduce the notation

$$\mathcal{R}_I \stackrel{\text{def}}{=} \left\{ y \in \mathbb{R}^m \mid \mathcal{F}_I(y) \neq \emptyset, \text{rank}(Y_I) < |I| \right\}$$

implying that if $\mathcal{R}_I \neq \emptyset$, the matrix Y_I is degenerate in the sense that it has at least one degenerate row, which is not linearly independent of its other rows. In other words, there exists a linear subspace $T_I \subseteq \mathbb{R}^m$ of $\mathbb{R}^m$ (namely the one belonging to the degenerate row of Y_I) such that $\mathcal{R}_I \subseteq T_I$ and $\dim(T_I) \leq m - 1$. Now, observe that the set

$$\mathcal{R} \stackrel{\text{def}}{=} \mathbb{Y} \setminus \left(\bigcup_{I \subseteq \{1, \dots, m\}} \mathcal{R}_I \right)$$

corresponds—by definition—to the set of parameters y for which $P(Y, y)$ is an entirely simple polyhedron. Because $\mathbb{Y}$ has non-empty interior in $\mathbb{R}^m$ (see [Corollary 1](#)) and because the sets $\mathcal{R}_I$ are contained in the not full-dimensional subspaces T_I of $\mathbb{R}^m$, the set $\mathcal{R}$ is dense in $\mathbb{Y}$. Thus, if $\mathbb{Y}_{\mathcal{I}}$ is a regular configuration domain, the set $\mathcal{R} \cap \mathbb{Y}_{\mathcal{I}}$ is non-empty. That is, we can find a point $\sigma \in \mathbb{Y}_{\mathcal{I}}$ such that $P(Y, \sigma)$ is an entirely simple polyhedron. Note that the same argument implies that the statement of [Corollary 3](#) holds, as the complement of $\mathcal{R}$ in $\mathbb{Y}$ has Lebesgue measure zero.

Appendix B. Proof of Corollary 2

By definition, $P(Y, y)$ is locally configuration stable if and only if $y \in \text{int}(\mathbb{Y}_{\mathcal{C}(y)})$. Therefore, the “if” part of the corollary follows as in Part I of the above proof of [Theorem 1](#) by replacing σ by y . Concerning the “only if” part, note that if $\mathcal{F}_I(y)$ is non-empty, the equation $Y_I x_I = y_I$ is feasible. If Y_I has a degenerate row, there exist arbitrarily small perturbations of y for which this equation becomes infeasible and the face configuration changes. Thus, by using the definition of $\mathcal{R}$ from Part II of the above proof of [Theorem 1](#), we find that $\mathcal{R} \cap \mathbb{Y}_{\mathcal{C}(y)} \supseteq \text{int}(\mathbb{Y}_{\mathcal{C}(y)})$. Since $y \in \text{int}(\mathbb{Y}_{\mathcal{C}(y)})$, the definition of $\mathcal{R}$ implies that $P(Y, y)$ is entirely simple.

References

- Bemporad, A., Morari, M., Dua, V., & Pistikopoulos, E. (2002). The explicit linear quadratic regulator for constrained systems. *Automatica*, 38(1), 3–20.
- Bertsekas, D. P. (2012). *Dynamic programming and optimal control* (3rd ed.). Belmont, Massachusetts: Athena Scientific Dynamic Programming and Optimal Control.
- Bertsimas, D., & Goyal, V. (2012). On the power and limitations of affine policies in two-stage adaptive optimization. *Mathematical Programming*, 134(2), 491–531.
- Bitsoris, G. (1988). On the positive invariance of polyhedral sets for discrete-time systems. *Systems and Control Letter*, 11(3), 243–248.
- Blanchini, F., & Miani, S. (2015). *Systems & control: foundations & applications, Set-theoretic methods in control* (p. xvi+481). Boston, MA: Birkhäuser Boston, Inc.
- Boyd, S., & Vandenberghe, L. (2004). *Convex optimization*. Cambridge University Press.
- Chen, L. (2005). New analysis of the sphere covering problems and optimal polytope approximation of convex bodies. *Journal of Approximation Theory*, 133, 134–145.
- Chisci, L., Rossiter, J. A., & Zappa, G. (2001). Systems with persistent disturbances: predictive control with restricted constraints. *Automatica*, 37, 1019–1028.

- Diehl, M., & Bjornberg, J. (2004). Robust dynamic programming for min-max model predictive control of constrained uncertain systems. *IEEE Transactions on Automatic Control*, 49(12), 2253–2257.
- Federer, H. (1959). Curvature measures. *Transactions of the American Mathematical Society*, 93, 418–491.
- Fleming, J., Kouvaritakis, B., & Cannon, M. (2015). Robust tube MPC for linear systems with multiplicative uncertainty. *Transactions on Automatic Control*, 60(4), 1087–1092.
- Goulart, P. J., Kerrigan, E. C., & Maciejowski, J. M. (2006). Optimization over state feedback policies for robust control with constraints. *Automatica*, 42(4), 523–533.
- Grünbaum, B. (1967). *Convex polytopes*. John Wiley & Sons.
- Gutman, P. O., & Cwikel, M. (1986). Admissible sets and feedback control for discrete-time linear dynamical systems with bounded controls and states. *IEEE Transactions on Automatic Control*, 31(4), 373–376.
- Houska, B., & Müller, M. A. (2017). Cost-to-travel functions: a new perspective on optimal and model predictive control. *Systems & Control Letters*, 106, 79–86.
- Köhler, J., Andina, E., Soloperto, R., Müller, M. A., & Allgöwer, F. (2019). Linear robust adaptive model predictive control: Computational complexity and conservatism. In *2019 IEEE 58th conference on decision and control* (pp. 1383–1388).
- Kothare, V., Balakrishnan, V., & Morari, M. (1996). Robust constrained model predictive control using linear matrix inequalities. *Automatica*, 32(10), 1361–1379.
- Langson, W., Chrysochoos, I., Raković, S. V., & Mayne, D. Q. (2004). Robust model predictive control using tubes. *Automatica*, 40(1), 125–133.
- Mayne, D. Q. (2015). Robust and stochastic MPC: are we going in the right direction? *IFAC-PapersOnLine*, 48(23), 1–8.
- Mayne, D. Q., Seron, M. M., & Raković, S. (2005). Robust model predictive control of constrained linear systems with bounded disturbances. *Automatica*, 41(2), 219–224.
- McMullen, P. (1971). The maximum numbers of faces of a convex polytope. *Mathematika*, 17, 179–184.
- Raković, S. V., Dai, L., & X., Yuanqing (2022). Homothetic tube model predictive control for nonlinear systems. *IEEE Transactions on Automatic Control*, 16–19.
- Raković, S., Kouvaritakis, B., & Cannon, M. (2013). Equi-normalization and exact scaling dynamics in homothetic tube model predictive control. *Systems & Control Letters*, 62(2), 209–217.
- Raković, S. V., Kouvaritakis, B., Cannon, M., Panos, C., & Findeisen, R. (2011). Fully parameterized tube MPC. *IFAC Proceedings Volumes*, 44(1), 197–202.
- Raković, S. V., Kouvaritakis, B., Cannon, M., Panos, C., & Findeisen, R. (2012). Parameterized tube model predictive control. *Transactions on Automatic Control*, 57(11), 2746–2761.
- Raković, S., Kouvaritakis, B., Findeisen, R., & Cannon, M. (2012). Homothetic tube model predictive control. *Automatica*, 48(8), 1631–1638.
- Raković, S., Levine, W. S., & Açikmeşe, B. (2016). Elastic tube model predictive control. In *American control conference ACC*, 2016, (pp. 3594–3599). IEEE.
- Schneider, R. (1981). Zur optimalen Approximation konvexer Hyperflächen durch Polyeder. *Mathematische Annalen*, 256, 289–301.
- Scokaert, P. O. M., & Mayne, D. Q. (1998). Min-max feedback model predictive control for constrained linear systems. *IEEE Transactions on Automatic control*, 43(8), 1136–1142.
- Stanley, R. P. (1980). The number of faces of a simplicial convex polytope. *Advances in Mathematics*, 35, 236–238.
- Villanueva, M. E., De Lazzari, E., Müller, M. A., & Houska, B. (2020). A set-theoretic generalization of dissipativity with applications in tube MPC. *Automatica*, 122(109179).

- Villanueva, M. E., Quirynen, R., Diehl, M., Chachuat, B., & Houska, B. (2017). Robust MPC via min-max differential inequalities. *Automatica*, 77, 311–321.



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