



Convex operator-theoretic methods in stochastic control[☆]

Boris Houska

ShanghaiTech University, China

ARTICLE INFO

Article history:

Received 26 May 2023

Received in revised form 1 January 2025

Accepted 13 March 2025

Keywords:

Stochastic optimal control
Hamilton–Jacobi–Bellman equations
Fokker–Planck–Kolmogorov equations

ABSTRACT

This paper is about operator-theoretic methods for solving nonlinear stochastic optimal control problems to global optimality. These methods leverage on the convex duality between optimally controlled diffusion processes and Hamilton–Jacobi–Bellman (HJB) equations for nonlinear systems in an ergodic Hilbert–Sobolev space. In detail, a generalized Bakry–Emery condition is introduced under which one can establish the global exponential stabilizability of a large class of nonlinear systems. It is shown that this condition is sufficient to ensure the existence of solutions of the ergodic HJB for stochastic optimal control problems on infinite time horizons. Moreover, a novel dynamic programming recursion for bounded linear operators is introduced, which can be used to numerically solve HJB equations by a Galerkin projection.

© 2025 Elsevier Ltd. All rights are reserved, including those for text and data mining, AI training, and similar technologies.

1. Introduction

Bellman's principle of optimality (Bellman, 1957) and its dual counterpart, the Chapman–Kolmogorov equation (Chapman, 1928; Kolmogoroff, 1931) for controlled diffusion processes (Krylov, 2008), are the fundament of stochastic control theory. They describe the semigroup structure of the evolution of optimally operated stochastic processes. Their associated partial differential equations (PDEs) for continuous-time models are the well-known HJB equation (Fleming & Soner, 1993) and the controlled Fokker–Planck–Kolmogorov (FPK) equation (Bogachev, Krylov, Röckner, & Shaposhnikov, 2015).

While parabolic FPKs and HJBs are related to stochastic optimal control problems on finite time horizons, their elliptic counterparts model the evolution of optimally operated diffusion processes on an infinite horizon. One distinguishes between *ergodic optimal control* and *infinite horizon optimal control*.

- *Ergodic optimal control* is about the analysis of the limit behavior of optimally operated dynamic systems as well as the long-term average control performance. For instance, in the deterministic case, a control system might have optimal steady states or optimal periodic orbits, but it could as well be that it is optimal to stay on a homoclinic orbit or even to allow a chaotic limit behavior. Similarly, the optimal ergodic distribution of nonlinear stochastic control systems often has a highly complex multi-modal structure.

- *Infinite horizon optimal control* is about optimizing the system's transient behavior. Here, the hope is that this transient eventually converges to an optimal ergodic limit. For general nonlinear stochastic systems, however, such a convergence analysis often turns out to be difficult. This is because, in practice, one is often interested in optimizing the system's economic performance, which leads to indefinite stage cost functions. Unfortunately, it is often not even clear whether the integral over such indefinite costs has an expected value on an infinite time horizon.

The goal of the present paper is to address the above challenges in ergodic- and infinite horizon optimal control by building upon recent results in elliptic diffusion operator theory and functional analysis, reviewed below.

1.1. Literature review

The systematic mathematical study of stochastic systems goes back to the work of Kolmogoroff (1931), although related continuous-time electro-dynamic, molecular, quantum-theoretic, and thermal diffusions have been studied even earlier by Fokker (1914), von Smoluchowski (1915), Planck (1917), Chapman (1928). The relevance of such diffusions in the context of analyzing the ergodic behavior of dynamical systems has been observed by Has'minskiĭ (1960) in 1960, who found an elegant way of using Harnack's inequality in combination with a weak Lyapunov function in order to analyze the existence of steady states of FPK equations. And, in fact, it was only six years later that Wonham realized the great potential of Has'minskiĭ's work in the context of ergodic control theory (Wonham, 1966). By now, these historical developments are well-understood and have been extended by

[☆] The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Subhrakanti Dey under the direction of Editor Florian Dorfler.

E-mail address: boris.houska@gmx.de.

many authors, as reviewed in great detail in [Bogachev et al. \(2015\)](#) focusing on FPK equations and in [Krylov \(2008\)](#) focusing on controlled diffusion processes.

In contrast to linear FPKs, HJBs are nonlinear PDEs, which have been introduced by [Bellman \(1957\)](#). They model the backward-in-time evolution of the value function of either deterministic or stochastic optimal control problems. Here, the state of the HJB equation can be interpreted as the co-state of the FPK-constraint of certain optimal diffusion control problems. As such, HJBs and FPKs are closely related via the concept of duality in convex optimization ([Bismut, 1973](#); [Fleming & Vermes, 1989](#)). HJBs for optimal control of deterministic systems are often analyzed by introducing an artificial stochastic perturbation and then passing to the so-called viscosity limit by making this perturbation infinitesimally small. The corresponding viscosity methods go back to [Crandall, Ishii, and Lions \(1992\)](#). They can be considered as the main tool for analyzing parabolic HJBs on finite time horizons. The textbooks [Bardi and Capuzzo-Dolcetta \(1997\)](#) and [Fleming and Soner \(1993\)](#) offer a relatively complete overview of the associated literature on HJBs between 1957 and 1997.

Regarding the existence and properties of solutions to elliptic HJBs for ergodic and infinite-horizon optimal control, however, surprisingly little is known. In [Bardi and Capuzzo-Dolcetta \(1997\)](#) a whole chapter is dedicated to infinite horizon optimal control problems with exponentially discounted stage costs. The analysis methods in this chapter are, however, inapplicable to problems with a general indefinite stage cost. In the context of ergodic optimal control, this problem has partly been resolved by [Arisawa \(1998\)](#); [Arisawa & Lions, 1998](#)), who has proposed methods to analyze the limit of vanishing discounts. Moreover, [Barles \(Barles & Souganidis, 2000\)](#) as well as [Namah and Roquejoffre \(1999\)](#) have suggested methods for analyzing the long term behavior of HJBs with periodic Hamiltonians. These methods have in common that they focus on special classes of stochastic systems under periodicity assumptions. And, finally, yet another option for analyzing infinite horizon optimal control problems is to work with turnpikes ([Ašković, Trélat, & Zidani, 2022](#); [Trélat & Zuazua, 2015](#)), which can be used to analyze the long-term local behavior of systems near optimal steady-states or periodic orbits.

In contrast to ergodic HJBs, much more is known about the limit behavior of FPKs. This is not only due the above mentioned work by Has'minskii, but also due to relatively recent breakthroughs in functional analysis that can be traced back to the seminal work of [Bakry and Émery \(1985\)](#) on *carre-du-champ* operators. Based on earlier work by Gross on hypercontractivity and logarithmic Sobolev inequalities ([Gross, 1975](#)), [Arnold, Markowich, Toscani, and Unterreiter \(2001\)](#) were among the first to come up with a comprehensive and general analysis of the convergence of FPKs to ergodic limit states by featuring Bakry–Emery conditions. Even more practical conditions for deriving logarithmic Sobolev- and general entropy-type inequalities for diffusion semigroups and FPKs have appeared in an article by [Bolley and Gentil \(2010\)](#). The most recent developments regarding the trend to equilibrium of FPKs in the degenerate case are summarized in [Barbu \(2021\)](#), [Barbu and Röckner \(2023\)](#).

Finally, concerning recent developments in PDE-constrained optimization ([Hinze, Pinnau, Ulbrich, & Ulbrich, 2009](#)), optimal control of FPKs has received significant attention during the last five years ([Aronna & Tröltzsch, 2021](#); [Breiten, Kunisch, & Pfeiffer, 2018a, 2018b](#)). These articles focus, however, on least-squares type problems for FPKs that are not directly related to HJBs. The only exception is the work of [Bobkov, Gentil, and Ledoux \(2001\)](#), who are the first to attempt an analysis of Hamilton–Jacobi equations based on logarithmic Sobolev inequalities—even if these authors focus only on a very special class of HJBs that can be solved explicitly via the Hopf–Lax formula.

1.2. Contributions

This paper presents novel results about the existence and uniqueness of solutions to ergodic and infinite horizon stochastic optimal control problems. They are summarized, respectively, in [Theorems 1](#) and [2](#).

In detail, [Theorem 1](#) is about the existence of ergodic limits of optimally controlled nonlinear stochastic McKean–Vlasov systems on a general convex domain with reflecting boundary, assuming that a weak Has'minskii–Lyapunov function is available. Its proof is based on an energy dissipation argument for FPKs with mixed boundary conditions that models the evolution of the probability density function of its underlying nonlinear stochastic control system. The theorem is applicable to a general class of ergodic optimal control problems with indefinite stage cost and potentially multi-modal optimal limit distributions.

Similarly, [Theorem 2](#) is about the existence of weak solutions to ergodic HJBs with Neumann boundary condition in a suitably weighted Hilbert–Sobolev space and their relation to infinite horizon stochastic optimal control. It is based on a generalized Bakry–Emery condition for optimally controlled FPKs under which global exponential stabilizability can be established. This condition turns out to be sufficient for ensuring that the value function of the stochastic infinite horizon optimal control problem is a bounded linear operator, whose unique Riesz' representation can be used to construct a weak solution of the ergodic HJB.

A particularly relevant aspect of [Theorem 2](#) is that it succeeds in analyzing a class of nonlinear PDEs — in this case HJBs — by relying on convex functional analysis methods that have originally been designed for analyzing linear PDEs. This implies in particular that mature numerical methods for linear parabolic PDEs, such as finite element Galerkin methods, can be used to approximately solve finite- and infinite horizon stochastic optimal control problems via a discretized operator-theoretic dynamic programming recursion. Because such methods for discretizing linear parabolic PDEs are, in general, more accurate, easier to analyze, and easier to implement than methods for discretizing nonlinear PDEs, such as HJBs, operator-theoretic dynamic programming methods offer clear advantages over classical dynamic programming methods for optimal control.

1.3. Overview

This paper is structured as follows.

- Section [2](#) reviews the relations between controlled McKean–Vlasov systems and FPKs in the presence of reflecting boundary conditions.
- Section [3](#) explains how to use energy functionals to establish dissipativity results for optimally controlled FPKs, leading to the first main result on ergodic optimal control that is summarized in [Theorem 1](#).
- Section [4](#) proposes a novel approach to analyzing parabolic HJBs with Neumann boundary conditions in a suitably weighted Hilbert–Sobolev space.
- Section [5](#) introduces Lyapunov diffusion operators in order to analyze the stabilizability of controlled diffusions via a generalized Bakry–Emery condition.
- Section [6](#) introduces ergodic HJBs and presents the main result on infinite horizon optimal control that is summarized in [Theorem 2](#).
- Section [7](#) presents a finite element Galerkin projection based operator-theoretic dynamic programming method for solving HJBs, and two case studies.
- Section [8](#) concludes the paper.

1.4. Notation

Let $\Omega \subseteq \mathbb{R}^n$ denote an open set, $\overline{\Omega} = \text{cl}(\Omega)$ its closure, $\partial\Omega = \overline{\Omega} \setminus \Omega$ its boundary, and $\omega \in L^1_{\text{loc}}(\Omega)$ a strictly positive weight function. The sets

$$L^k(\Omega, \omega), C_0^\infty(\Omega), C^k(\overline{\Omega}), \text{ and } H^1(\Omega, \omega)$$

denote – as usual – the set of functions that are, respectively, L^k -integrable with respect to the weight ω , smooth with compact support, k -times continuously differentiable with continuous extensions to the boundary, and L^2 -integrable with L^2 -integrable weak derivatives on Ω , both with respect to the weight ω . Additionally,

$$(\varphi, \psi)_{L^2(\Omega, \omega)} \stackrel{\text{def}}{=} \int_{\Omega} \varphi \psi \omega \, dx$$

$$\text{and } (\varphi, \psi)_{H^1(\Omega, \omega)} \stackrel{\text{def}}{=} \int_{\Omega} (\varphi \psi + \nabla \varphi^\top \nabla \psi) \omega \, dx$$

are defined such that $L^2(\Omega, \omega)$ and $H^1(\Omega, \omega)$ are Hilbert spaces. Whenever it is clear from the context what Ω is, we denote these spaces by $L^2(\omega)$ or $H^1(\omega)$. Moreover, for the special case $\omega = 1$, we write $L^2(\Omega)$ or $H^1(\Omega)$. Here, ∇ denotes the gradient operator. For given time intervals $[0, T] \subseteq \mathbb{R}$ and any Banach space $(X, \|\cdot\|_X)$, we use the symbol $L^k(0, T; X)$ to denote the set of strongly measurable $\varphi : [0, T] \rightarrow X$ with

$$\|\varphi\|_{L^k(0, T; X)} \stackrel{\text{def}}{=} \left(\int_0^T \|\varphi(t)\|_X^k \, dt \right)^{1/k} < \infty,$$

such that $L^k(0, T; X)$ is a Bochner space. Weak derivatives with respect to time are indicated by a “dot”; and $H^1(\omega)^*$ denotes the dual of $H^1(\omega)$. The space

$$\mathcal{P}(\omega) \stackrel{\text{def}}{=} \{ \varphi \in L^2(0, T; H^1(\omega)) \mid \dot{\varphi} \in L^2(0, T; H^1(\omega)^*) \}$$

is then again a Hilbert space with respect to its natural inner product (Hinze et al., 2009, Thm. 1.31). The duality pairing between a Hilbert space and its dual is denoted by $\langle \cdot, \cdot \rangle$.

2. Stochastic control systems

This section is about the equivalence of closed-loop controlled stochastic nonlinear systems in a finite dimensional space and a special class of open-loop controlled deterministic linear systems in an infinite dimensional space, namely, controlled FPKs.

2.1. Nonlinear control systems

Deterministic control-affine systems have the form

$$\dot{x} = f(x) + G(x)u \quad \text{on } \mathbb{T} \stackrel{\text{def}}{=} (0, T). \quad (1)$$

Here, $\mathbb{T} \subseteq \mathbb{R}$ denotes an open time horizon. It is finite if $T < \infty$ and infinite if $T = \infty$. The right-hand functions $f \in C^1(\overline{\Omega})^{n_x}$ and $G \in C^1(\overline{\Omega})^{n_u}$ are assumed to be defined on the closure of an open domain $\Omega \subseteq \mathbb{R}^{n_x}$. Moreover, $x(t)$ denotes the state and $u(t)$ the control.

The goal is to design a feedback $\mu \in L^1_{\text{loc}}(0, T; L^1_{\text{loc}}(\Omega))$ that can be used to control (1) in closed-loop mode,

$$\dot{x}(t) = F[\mu](t, x(t)) \quad \text{for } t \in \mathbb{T}, \quad (2)$$

with $F[\mu](t, x) \stackrel{\text{def}}{=} f(x) + G(x)\mu(t, x)$. Solutions to (2) are defined in the viscosity sense (Crandall et al., 1992).

2.2. Stochastic differential equations

Let us temporarily assume that $\Omega = \mathbb{R}^{n_x}$. In this case, we can add a process noise to the right-hand of (2), arriving at the stochastic differential equation (SDE) (Oksendal, 2000)

$$dX = F[\mu](t, X) dt + \sqrt{2\epsilon} dW_t \quad \text{on } \mathbb{T}, \quad (3)$$

where W_t , $t \geq 0$, denotes a Wiener process, $\epsilon > 0$ a diffusion constant, and $x_0 \in \mathbb{R}^{n_x}$ an initial state. The state of this SDE at time $t \in \mathbb{T}$ is denoted by $X(t)$. It is a random variable with density function $\rho(t) \in L^1(\Omega)$. This means that for any given Borel set $S \subseteq \Omega$, the probability of the event $X(t) \in S$ is given by $\Pr(X(t) \in S) = \int_S \rho(t) dx$. Conditions under which (3) admits a martingale solution X that has such an L^1 -integrable probability density function ρ are discussed below.

2.3. Reflections at the boundary

The states of many nonlinear systems have no physical interpretation outside of their natural domain Ω . For instance the concentration of the reactants in a controlled chemical process are usually required to be non-negative implying that $\Omega \neq \mathbb{R}^{n_x}$. In such a case, the stochastic process noise model in (3) makes no sense, as the Wiener process W_t is unbounded. Instead, at least for the case that Ω is convex, a physically more meaningful stochastic model is given by

$$\sqrt{2\epsilon} dW_t \in dX - F[\mu](t, X)dt + N_{\overline{\Omega}}(X)dt \quad \text{on } \mathbb{T}. \quad (4)$$

This is a McKean–Vlasov SDE with reflection (Ciotir & Fayad, 2022; Sznitman, 1984), where the additional drift term,

$$N_{\overline{\Omega}}(x) \stackrel{\text{def}}{=} \{z \in \mathbb{R}^{n_x} \mid \forall y \in \overline{\Omega}, z^\top(x - y) \geq 0\},$$

acts as reflector. We have $N_{\overline{\Omega}}(x) = \{0\}$ for $x \in \Omega$, which means that (3) and (4) coincide on Ω . For $x \in \partial\Omega$, however, $N_{\overline{\Omega}}(x)$ is equal to the normal cone of $\overline{\Omega}$ at x . Consequently, whenever (4) admits a solution X that has an L^1 -integrable probability density ρ , we expect that

$$\Pr(X(t) \in \Omega) = \int_{\Omega} \rho(t) dx = 1, \quad (5)$$

which is the case under very mild assumptions that are, for instance, discussed in Ciotir and Fayad (2022), but sufficient conditions under which this is the case are also discussed below. For the special case that $\Omega = \mathbb{R}^{n_x}$, (4) reduces to (3).

2.4. Fokker–Planck–Kolmogorov equations

Let $\mathcal{A}$ denote the ϵ -diffusion operator of f and $\mathcal{B}$ the linear transport operator of G , formally defined by

$$\mathcal{A}\rho = -\text{div}(f\rho) + \epsilon \Delta \rho \quad \text{and} \quad \mathcal{B}v = -\text{div}(Gv),$$

where “div” denotes the divergence and “ Δ ” the Laplace operator. The pair $(\mathcal{A}, \mathcal{B})$ is associated with an infinite-dimensional linear control system, called the controlled FPK equation. In the presence of reflecting boundary conditions, it has the form

$$\begin{aligned} \dot{\rho} &= \mathcal{A}\rho + \mathcal{B}v & \text{on } \Omega_{\mathbb{T}} \\ \rho_0 &= \rho(0) & \text{on } \Omega \\ 0 &= (\epsilon \nabla \rho - f\rho - Gv)^\top \eta|_{\partial\Omega} & \text{on } \Sigma_{\mathbb{T}}. \end{aligned} \quad (6)$$

Here, $\rho_0 \in L^1(\Omega)$ denotes a given initial probability distribution on Ω , while $\Omega_{\mathbb{T}} \stackrel{\text{def}}{=} \mathbb{T} \times \Omega$ and $\Sigma_{\mathbb{T}} \stackrel{\text{def}}{=} \mathbb{T} \times \partial\Omega$ denote, respectively, the FPK’s open domain and its spatial boundary. Moreover, $\eta : \partial\Omega \rightarrow \mathbb{R}^{n_x}$ denotes an outer normal, such that $\eta(x) \in N_{\overline{\Omega}}(x)$ and $\|\eta(x)\|_2 = 1$ for all $x \in \partial\Omega$. The open-loop linear control system (6) is closely related to the closed-loop nonlinear control

system (4) after identifying ρ with the probability distribution of the random process X and after substituting

$$v = \mu\rho \quad \text{on } \Omega_{\mathbb{T}}. \quad (7)$$

In order to elaborate on this relation between (4), (6) and (7), we define $\mathcal{U}(\omega) \stackrel{\text{def}}{=} (L^2(0, T; L^2(\omega)))^{n_u}$, for $\omega \in L^1_{\text{loc}}(\Omega)$, $\omega > 0$. The linear operators $\mathcal{A}$ and $\mathcal{B}$ are associated with the bilinear forms

$$a(\rho, V) \stackrel{\text{def}}{=} \int_{\Omega} \rho f^{\top} \nabla V - \epsilon \nabla \rho^{\top} \nabla V \, dx$$

$$\text{and} \quad b(v, V) \stackrel{\text{def}}{=} \sum_{i=1}^{n_u} \langle v_i, \nabla V^{\top} G_i \rangle,$$

which are defined for all $\rho \in \mathcal{P}(\omega^{-1})$ and all $V \in \mathcal{P}(\omega)$.

Definition 1. Let $\omega \in L^{\infty}(\Omega)$ with $\omega > 0$ on Ω and $v \in \mathcal{U}(\omega^{-1})$ be given. A function $\rho \in \mathcal{P}(\omega^{-1})$ with given initial value $\rho(0) = \rho_0 \in L^2(\omega^{-1})$ is called a weak solution (Hinze et al., 2009, Def. 1.27) of (6) if

$$\int_0^T \langle \dot{\rho}, V \rangle \, dt = \int_0^T a(\rho, V) + b(v, V) \, dt \quad (8)$$

for all test functions $V \in \mathcal{P}(\omega)$.

For the special case that Ω is convex and bounded and $T < \infty$, we may set the weight function to $\omega = 1$. In this case, the existence of a unique weak solution of (6) for arbitrary $v \in \mathcal{U}(1)$ follows from a standard result about parabolic PDEs, see Hinze et al. (2009, Thm. 1.33), recalling that f and G are assumed to be continuously differentiable on $\overline{\Omega}$ and $\epsilon > 0$. For unbounded domains $\Omega_{\mathbb{T}}$, however, additional assumptions on f , G , and ω are needed in order to ensure the existence of weak solutions, as discussed in Section 3.

2.5. Nonlinear control via linear systems theory

For the purpose of analyzing the finite-dimensional closed-loop controlled nonlinear system (f, G) , as defined by (4), it is sufficient to analyze the infinite dimensional open-loop controlled linear system $(\mathcal{A}, \mathcal{B})$, as defined in (6), see also Korda and Mezić (2018), Lasserre, Henrion, Prieur, and Trelat (2008), Villanueva, Jones, and Houska (2021), Vinter (1993). This is because the existence of a unique weak solution $\rho \in \mathcal{P}(\omega^{-1})$ of (6) for $\omega \in L^{\infty}(\Omega)$ and an initial probability density function $0 \leq \rho_0 \in L^2(\omega^{-1}) \cap L^1(\Omega)$ with $\int_{\Omega} \rho_0 \, dx = 1$ of the initial random variable $X(0)$ is sufficient to ensure that (4) has a martingale solution X with probability density function ρ that satisfies (5). This is a standard result from the theory of diffusion processes (Trevisan, 2016), which has recently been extended for general FPKs with reflecting boundary conditions (Barbu, 2021); see also Ciotir and Fayad (2022). Of course, here we have to substitute (7), but this substitution is justified, too. Namely, it follows from the minimum principle for parabolic FPKs (Bogachev et al., 2015) that if $\rho \in \mathcal{P}(\omega^{-1})$, is a weak solution of (6) for an initial probability density $\rho_0 \geq 0$, then

$$\rho > 0 \quad \text{on } \Omega_{\mathbb{T}}, \quad (9)$$

recalling that $\Omega_{\mathbb{T}}$ is an open set, $0 \notin \mathbb{T}$, and $\epsilon > 0$. Thus, the closed-loop control law, $\mu = v/\rho$, can be recovered.

3. Ergodic optimal control

The idea to analyze optimal control of diffusion processes with reflection can be traced back to the work of Puterman (1977), who originally introduced such optimal control problems on a finite time horizon $\mathbb{T}$ and on a bounded domain Ω . This section presents methods for analyzing such optimal control problems on potentially unbounded domains, with a focus on ergodic limits.

3.1. Problem formulation

Let $\mathbb{U} \subseteq \mathbb{R}^{n_u}$ denote a control constraint set, let

$$\mathbb{U} \stackrel{\text{def}}{=} \{\mu \in L^1_{\text{loc}}(0, T; L^1_{\text{loc}}(\Omega)) \mid \mu \in \mathbb{U} \text{ on } \Omega_{\mathbb{T}}\}$$

denote the associated set of feasible control laws, and let $\ell : \Omega \times \mathbb{U} \rightarrow \mathbb{R}$ be a cost function. The current section is about ergodic optimal control problems of the form

$$J^* = \lim_{T \rightarrow \infty} \min_{\mu \in \mathbb{U}} \mathbb{E} \left\{ \frac{1}{T} \int_0^T \ell(X(t), \mu(X(t))) \, dt \right\}. \quad (10)$$

Here, X denotes a martingale solution of SDE (4) that depends implicitly on the optimization variable μ .

Assumption 1. We assume that $\epsilon > 0$ and that

1. the set $\Omega \subseteq \mathbb{R}^{n_x}$ is convex, open, and non-empty;
2. the system satisfies $(f, G) \in C^1(\overline{\Omega})^{n_x \times (1+n_u)}$;
3. the set $\mathbb{U} \subseteq \mathbb{R}^{n_u}$ is convex, compact, $\mathbb{U} \neq \emptyset$; and
4. the stage cost $\ell \in C^2(\overline{\Omega} \times \mathbb{U})$ is bounded from below, radially unbounded in x ,

$$\|x\| \rightarrow \infty \implies \ell(x, u) \rightarrow \infty,$$

and strongly convex in u , $\nabla_u^2 \ell > 0$ on $\Omega \times \mathbb{U}$.

Assumption 1 neither excludes that Ω is unbounded nor that $\Omega = \mathbb{R}^{n_x}$. On such unbounded domains, however, a weak Has'minskiĭ-Lyapunov function is needed.

Assumption 2. There exists a function $Q \in C^2(\overline{\Omega})$ and positive constants $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in (0, \infty)$ such that

1. the Hessian of Q is positive definite and uniformly bounded, such that

$$\gamma_1 I \leq \nabla^2 Q \leq \gamma_2 I \quad \text{on } \Omega, \quad (11)$$

2. the function Q satisfies $\nabla Q^{\top} \eta \geq 0$ on $\partial\Omega$, and
3. Q satisfies the weak Lyapunov condition

$$F[\mu]^{\top} \nabla Q \leq \gamma_3 - \gamma_4 \ell[\mu] \quad \text{on } \Omega \quad (12)$$

$$\text{with} \quad \ell[\mu](x) \stackrel{\text{def}}{=} \ell(x, \mu(x)) \quad (13)$$

for at least one $\mu \in C^1(\Omega)$ with $\mu \in \text{int}(\mathbb{U})$ on Ω .

Assumption 2 can be interpreted as a weak stabilizability condition of the system (f, G) relative to its cost ℓ . In contrast to Has'minskiĭ's original condition for general parabolic PDEs (Has'minskiĭ, 1960), however, this assumption is tailored to our optimal control problem setting.

Example 1. If Assumption 1 holds and if the domain $\Omega = \{x \mid \|x\| < 1\}$ is the open unit ball, Assumption 2 is always satisfied, because Condition (12) holds with

$$Q(x) = \frac{1}{2} \|x\|^2 \quad \text{and} \quad \gamma_1 = \gamma_2 = \gamma_4 = 1$$

for all sufficiently large constants γ_3 .

Example 2. Let $f(x) = x - x^3$, $G(x) = 0$, and the cost $\ell(x, u) = x^2 + x^4 + u^2$ be defined on the unbounded domain $\Omega = \mathbb{R}$. In this case, Assumption 2 is satisfied for $\mu = 0$,

$$Q(x) = \frac{1}{2} \|x\|^2, \quad \gamma_1 = \gamma_2 = \gamma_3 = 1, \quad \text{and} \quad \gamma_4 = \frac{1}{4}.$$

The associated nominal nonlinear system, however, is neither controllable nor stabilizable and, consequently, does not admit a strong control Lyapunov function.

The above examples suggest that [Assumptions 1](#) and [2](#) are satisfied for a large class of nonlinear control systems. This statement is additionally supported by a long list of examples for generic FPKs in [Bogachev et al. \(2015\)](#), where a very similar definition of Has'minskii–Lyapunov functions is used.

3.2. Optimal steady states

This section discusses methods for analyzing the existence of optimal steady state of the FPK.

Lemma 1. *Let [Assumptions 1](#) and [2](#) be satisfied and let $\mu \in C^1(\Omega)$ be a control law for which [\(12\)](#) holds. Then there exists an equilibrium $\rho_s \in H^1(\Omega) \cap L^1(\Omega)$ such that*

$$\begin{aligned} 0 &= \mathcal{A}\rho_s + \mathcal{B}(\mu\rho_s) && \text{on } \Omega, \\ 0 &\geq \rho_s && \text{on } \Omega, \\ 0 &= (\epsilon \nabla \rho_s - F[\mu]\rho_s)^\top \eta && \text{on } \partial\Omega, \\ 1 &= \int_{\Omega} \rho_s(x) dx, \\ \frac{\epsilon\gamma_2 n_x + \gamma_3}{\gamma_4} &\geq \int_{\Omega} \ell(x, \mu(x))\rho_s(x) dx. \end{aligned} \quad (14)$$

Proof. Let Q be defined as in [Assumption 2](#) and let

$$\Omega_\alpha \stackrel{\text{def}}{=} \{x \in \Omega \mid Q(x) < \alpha\} \quad \text{for } \alpha > \inf_{x \in \Omega} Q(x)$$

denote the associated open α -sublevel sets of Q . Because Q is strongly convex and differentiable and because α is assumed to be sufficiently large, the sets Ω_α are open, non-empty, convex, bounded, and have a Lipschitz boundary. Thus, it makes sense to introduce the convex PDE-constrained auxiliary optimization problem

$$\begin{aligned} \tilde{\ell}_\alpha &\stackrel{\text{def}}{=} \sup_{\rho} \int_{\Omega_\alpha} \gamma_4 \ell[\mu]\rho dx \\ \text{s.t. } \begin{cases} 0 &= \epsilon \Delta \rho - \text{div}(F[\mu]\rho) && \text{on } \Omega_\alpha \\ 0 &\geq \rho && \text{on } \Omega_\alpha \\ 0 &= (\epsilon \nabla \rho - F[\mu]\rho)^\top \eta_\alpha && \text{on } \partial\Omega_\alpha, \\ 1 &= \int_{\Omega_\alpha} \rho dx \end{cases} \end{aligned} \quad (15)$$

with optimization variable $\rho \in H^1(\Omega_\alpha) \cap L^1(\Omega_\alpha)$. Here, η_α denotes an outer normal of Ω_α . Note that the generalized version of Has'minskii's theorem on the existence of unique probability solutions of the stationary FPK on convex domains ([Huang, Ji, Liu, & Yi, 2015](#), Thm. A), see also [Bogachev et al. \(2015\)](#), implies that [\(15\)](#) is feasible. Since we assume that ℓ is bounded from below, it follows that there exists a $\underline{\sigma} > -\infty$ with $\underline{\sigma} < \tilde{\ell}_\alpha$. Moreover, a Lagrangian relaxation of [\(15\)](#) yields

$$\begin{aligned} \tilde{\ell}_\alpha &\leq \sup_{\rho \geq 0} \left\{ \bar{\sigma} + (\gamma_4 \ell[\mu] + F[\mu]^\top \nabla Q - \bar{\sigma}, \rho)_{L^2(\Omega_\alpha)} \right. \\ &\quad \left. - \epsilon (\nabla Q^\top, \nabla \rho)_{L^2(\Omega_\alpha)} \right\} \\ &= \sup_{\rho \geq 0} \left\{ \bar{\sigma} + (\gamma_4 \ell[\mu] + F[\mu]^\top \nabla Q + \epsilon \Delta Q - \bar{\sigma}, \rho)_{L^2(\Omega_\alpha)} \right. \\ &\quad \left. - \epsilon \int_{\partial\Omega} \rho \nabla Q^\top \eta_\alpha dx \right\} \leq \bar{\sigma}, \end{aligned} \quad (16)$$

for sufficiently large constants $\bar{\sigma} > 0$. The latter inequality follows from [\(11\)](#), [\(12\)](#), and the fact that

$$\nabla Q^\top \eta \geq 0 \text{ on } \partial\Omega \implies \nabla Q^\top \eta_\alpha \geq 0 \text{ on } \partial\Omega_\alpha$$

holds due to the above construction of Ω_α . More precisely, [\(11\)](#) and [\(12\)](#) imply that

$$\Delta Q \leq n_x \gamma_2 \quad \text{and} \quad \gamma_4 \ell[\mu] + F[\mu]^\top \nabla Q \leq \gamma_3.$$

Consequently, [\(16\)](#) holds for $\bar{\sigma} = \gamma_3 + \epsilon n_x \gamma_2$. But this particular choice for $\bar{\sigma}$ does not depend on α . Thus, the bounds $\underline{\sigma} \leq \ell_\alpha \leq \bar{\sigma}$

hold uniformly. The statement of the theorem follows then by taking the limit $\alpha \rightarrow \infty$ in [\(15\)](#) and dividing the upper bound on $\tilde{\ell}_\infty$ by γ_4 . $\square$

The stage cost of [\(10\)](#) can be represented by the convex perspective function

$$\mathfrak{L}(\rho, v) \stackrel{\text{def}}{=} \int_{\Omega} \ell\left(\cdot, \frac{v}{\rho}\right) \rho dx. \quad (17)$$

It is defined for any strictly positive and measurable function $\rho > 0$ and any measurable function v for which the above integral exists. Similarly, control constraints are represented by the convex cone

$$\mathcal{K} \stackrel{\text{def}}{=} \left\{ (\rho, v) \in H^1(\Omega) \times L^2(\Omega)^{n_u} \mid \begin{array}{ll} v \in \rho \mathbb{U} & \text{on } \Omega \\ \rho \geq 0 & \text{on } \Omega \end{array} \right\}.$$

Thus, optimal steady states can be found by solving the convex optimization problem

$$\begin{aligned} \ell_\infty &\stackrel{\text{def}}{=} \min_{(\rho, v) \in \mathcal{K}} \mathfrak{L}(\rho, v) \\ \text{s.t. } \begin{cases} 0 &= \mathcal{A}\rho + \mathcal{B}v && \text{on } \Omega \\ 0 &= (\epsilon \nabla \rho - f\rho - Gv)^\top \eta && \text{on } \partial\Omega, \\ 1 &= \int_{\Omega} \rho dx, \end{cases} \end{aligned} \quad (18)$$

with $(\rho, v) \in \mathcal{K}$ denoting optimization variables.

Corollary 1. *Let [Assumptions 1](#) and [2](#) be satisfied. Then [\(18\)](#) has a unique minimizer $(\rho_\infty, v_\infty) \in \mathcal{K}$.*

Proof. The construction of ρ_s and μ in [Lemma 1](#) satisfies

$$(\rho_s, \mu\rho_s) \in \text{int}(\mathcal{K}).$$

As such, the closed and convex cone $\mathcal{K}$ is compatible with Slater's constraint qualification, recalling that $\rho_s > 0$ follows from the minimum principle for elliptic operators ([Bogachev et al., 2015](#)). Consequently, feasibility and boundedness of [\(18\)](#) follows immediately from [Lemma 1](#), as $(\rho, v) = (\rho_s, \mu\rho_s)$ is a feasible point with bounded objective value. In summary, [\(18\)](#) has the following properties:

- (1) Slater's (weak) constraint qualification holds,
- (2) the objective function $\mathcal{L}$ is a convex perspective function that is – due to [Assumption 1](#) – strongly convex in the control v , and
- (3) the operator $\mathcal{A}$ is uniformly elliptic.

Consequently, the statement of this corollary follows from a well-known standard result on the existence and uniqueness of minimizers of elliptic PDE-constrained convex optimization problems in Hilbert–Sobolev spaces; see, for example, [Hinze et al. \(2009, Thm 1.45\)](#). $\square$

3.3. Dissipation of energy

Let (ρ_∞, v_∞) denote the optimal steady state, as defined in [Corollary 1](#), and let $\mu_\infty = v_\infty/\rho_\infty$ denote an associated optimal control law. It can be used to study the controlled parabolic FPK

$$\begin{aligned} \dot{\rho} &= \mathcal{A}\rho + \mathcal{B}(\mu_\infty \rho) && \text{on } \Omega_\mathbb{T} \\ \rho_0 &= \rho(0) && \text{on } \Omega \\ 0 &= (\epsilon \nabla \rho - F[\mu_\infty]^\top \rho)^\top \eta|_{\partial\Omega} && \text{on } \Sigma_\mathbb{T} \end{aligned} \quad (19)$$

on a possibly infinite time horizon $\mathbb{T}$. Here, $\rho_0 \in \mathcal{P}_0(\rho_\infty^{-1})$ denotes an initial probability density, where

$$\mathcal{P}_0(\rho_\infty^{-1}) \stackrel{\text{def}}{=} \left\{ \rho_0 \in L^2(\rho_\infty^{-1}) \cap L^1(\Omega) \mid \int_{\Omega} \rho_0 dx = 1, \rho_0 \geq 0 \text{ on } \Omega \right\}.$$

In this setting, the key idea for analyzing (19) is to introduce the energy functional

$$E \stackrel{\text{def}}{=} \|\rho - \rho_\infty\|_{L^2(\rho_\infty^{-1})}^2 = \int_{\Omega} \frac{(\rho - \rho_\infty)^2}{\rho_\infty} dx.$$

Its weak time derivative is given by

$$\frac{dE}{dt} = -2\epsilon \int_{\Omega} \left\| \nabla \left(\frac{\rho}{\rho_\infty} \right) \right\|_2^2 \rho_\infty dx \leq 0. \quad (20)$$

Note that this equation is well-known in the functional analysis and FPK literature (Arnold et al., 2001; Bogachev et al., 2015), where E is often interpreted as energy (or entropy) that is dissipated during the evolution of the FPK. Alternatively, by translating this to control engineering language, E could also be called a Lyapunov function for (19).

Lemma 2. Let Assumptions 1 and 2 be satisfied. Then, (19) has for every $\rho_0 \in \mathcal{P}_0(\rho_\infty^{-1})$ a unique solution

$$\rho \in \mathcal{P}(\rho_\infty^{-1}) \quad \text{with} \quad \begin{cases} \rho > 0 & \text{on } \Omega_{\mathbb{T}} \\ \int_{\Omega} \rho dx = 1 & \text{on } \mathbb{T}. \end{cases}$$

Moreover, ρ satisfies (20). This results holds on finite and infinite time horizons $\mathbb{T}$.

Proof. FPKs have been analyzed by so many authors that is impossible to list them all. As such, Arnold et al. (2001) and Bogachev et al. (2015) should be regarded as selected examples for two out of a long list of articles in which proofs of variants of the current lemma can be found. Nevertheless, because this is important for the developments in this paper, it appears appropriate to recall the following three main arguments why the above statement holds.

(1) First, if $\Omega_{\mathbb{T}}$ is bounded, a unique weak solution $\rho \in \mathcal{P}(\rho_\infty^{-1})$ of (19) exists, since it is a uniformly parabolic linear PDE. Moreover, Harnack's inequality yields

$$\phi \stackrel{\text{def}}{=} \frac{\rho}{\rho_\infty} \in H^1(\Omega, \rho_\infty) \quad \text{on } \mathbb{T}. \quad (21)$$

The details of this argument can be found in Bogachev et al. (2015, Chapter 7), where the square-integrability of logarithmic gradients of FPKs is discussed in all detail.

(2) Next, (20) can be verified by direct computation,

$$\begin{aligned} \frac{dE}{dt} &= 2 \langle \dot{\rho}, \phi - 1 \rangle = 2 \langle \dot{\rho}, \phi - 1 \rangle - \langle \dot{\rho}_\infty, \phi^2 \rangle \\ &\stackrel{(19)}{=} 2 \int_{\Omega} (\rho F[\mu_\infty]^\top \nabla \phi - \epsilon \nabla \rho^\top \nabla \phi) dx \\ &\quad - 2 \int_{\Omega} [\rho_\infty F[\mu_\infty]^\top \nabla \phi - \epsilon \nabla \rho_\infty^\top \nabla \phi] \phi dx \\ &\stackrel{(21)}{=} -2\epsilon \int_{\Omega} \|\nabla \phi\|_2^2 \rho_\infty dx \leq 0, \end{aligned} \quad (22)$$

where the equation in the second line follows by using that ρ_∞ satisfies (19) with $\rho_0 = \rho_\infty$ and $\dot{\rho}_\infty = 0$. Thus, Gronwall's lemma yields the a-priori energy estimate

$$\|\rho(t) - \rho_\infty\|_{L^2(\rho_\infty^{-1})} \leq \|\rho_0 - \rho_\infty\|_{L^2(\rho_\infty^{-1})} \quad \text{on } \mathbb{T}. \quad (23)$$

(3) For the case that Ω and $\mathbb{T}$ are unbounded, one can proceed as in the proof of Lemma 1. This means that the above two steps are repeated in order to construct a sequence of solutions $\rho_k \in \mathcal{P}(\rho_\infty^{-1})$ of (19) on the bounded domains $\Omega_{\alpha_k} \times (0, T_k)$ with increasing sequences $\alpha_k \rightarrow \infty$ and $T_k \rightarrow \infty$. Because the energy estimate (23) yields a uniform upper bound, the sequence $(\rho_k)_{k \in \mathbb{N}}$ has a weakly convergent subsequence in the Hilbert space $\mathcal{P}(\rho_\infty^{-1})$, whose limit must be a weak solution of (19) and, by continuity, (20) holds, too.

In addition to the above three arguments, we recall that FPKs model the evolution of probability densities and, as such, the conservation law $\int_{\Omega} \rho dx = \int_{\Omega} \rho_0 dx = 1$ holds on $\mathbb{T}$, referring to Bogachev et al. (2015) for a formal discussion of this property. This completes the proof of the lemma. $\square$

3.4. Optimal ergodic limits

In order to analyze the ergodic optimal control problem (10), an additional assumption is introduced.

Assumption 3. We have $\ell[\mu_\infty] \in L^2(\rho_\infty)$.

This assumption is equivalent to requiring that $\ell[\mu_\infty]$ has a variance at the steady state ρ_∞ ; that is,

$$\int_{\Omega} \ell[\mu_\infty]^2 \rho_\infty dx - \ell_\infty^2 < \infty.$$

Thus, if Assumptions 1 and 2 hold and if Ω is bounded, then Assumption 3 automatically holds. Otherwise, if Ω is unbounded, such a variance exists under very mild growth conditions on ℓ , as discussed in all detail in Bogachev et al. (2015, Chapter 3), where upper bound estimates for ρ_∞ can be found. Next, let

$$\begin{aligned} J(T, \rho_0) &\stackrel{\text{def}}{=} \min_{(\rho, v) \in \mathcal{K}_{\mathbb{T}}} \int_0^T \mathcal{L}(\rho, v) dt \\ \text{s.t.} \quad &\begin{cases} \dot{\rho} = \mathcal{A}\rho + \mathcal{B}v & \text{on } \Omega_{\mathbb{T}} \\ 0 = (\epsilon \nabla \rho - f\rho - Gv)^\top \eta & \text{on } \Sigma_{\mathbb{T}} \\ \rho_0 = \rho(0) & \text{on } \Omega \end{cases} \end{aligned} \quad (24)$$

denote the value function of the stochastic control problem on the finite time horizon $\mathbb{T} = (0, T)$ with parameters $T > 0$ and $\rho_0 \in \mathcal{P}_0(\rho_\infty^{-1})$. Here, the optimization variables ρ and v are elements of the convex cone

$$\mathcal{K}_{\mathbb{T}} \stackrel{\text{def}}{=} \left\{ (\rho, v) \left| \begin{array}{l} \rho \in \mathcal{P}(\rho_\infty^{-1}) \\ v \in \mathcal{U}(\rho_\infty^{-1}) \\ (\rho(t), v(t)) \in \mathcal{K} \quad \text{f.a. } t \in \mathbb{T} \end{array} \right. \right\}.$$

Problem (10) is equivalent to computing the limit

$$J^* = \lim_{T \rightarrow \infty} \frac{J(T, \rho_0)}{T}. \quad (25)$$

The following theorem proves that $J^* = \ell_\infty$, which, in turn, implies that J^* does not depend on ρ_0 .

Theorem 1. Let Assumptions 1, 2, and 3 be satisfied and let ℓ_∞ be defined as in (18). Then (24) has for all $T > 0$ and all $\rho_0 \in \mathcal{P}(\rho_\infty^{-1})$ a unique minimizer $(\rho^*, v^*) \in \mathcal{K}_{\mathbb{T}}$. Moreover, the ergodic limit (25) exists. It is given by

$$\forall \rho_0 \in \mathcal{P}_0(\rho_\infty^{-1}), \quad \lim_{T \rightarrow \infty} \frac{J(T, \rho_0)}{T} = \ell_\infty.$$

Proof. This proof is divided into two parts, where the first part verifies the claim that (24) has a unique minimizer while the second part is about the ergodic limit.

Part I. First note that an upper bound on the stage cost of (24) can be found by applying the triangle- and the Cauchy-Schwarz inequality,

$$\begin{aligned} |\mathcal{L}(\rho, v)| &\leq \ell_\infty + |\mathcal{L}(\rho, v) - \ell_\infty| \\ &= \ell_\infty + \left| \int_{\Omega} \ell[\mu_\infty](\rho - \rho_\infty) dx \right| \\ &\leq \ell_\infty + \|\rho - \rho_\infty\|_{L^2(\rho_\infty^{-1})} \|\ell[\mu_\infty]\|_{L^2(\rho_\infty)}, \end{aligned} \quad (26)$$

which holds along the trajectory $(\rho, v) = (\rho, \mu_\infty \rho)$ of (19). Due to Assumption 3 and the energy estimate (23) from the proof of Lemma 2, this upper bound is sufficient to ensure that

$J(T, \rho_0) < \infty$. Thus, the first part of this proof is analogous to the argument in the proof of [Corollary 1](#): the conic constraint $(\rho, v) \in \mathcal{K}_{\mathbb{T}}$ is strictly feasible, which implies that Slater's weak constraint qualification holds, and $\mathcal{L}$ is strictly convex in v . Consequently, the existence of a unique minimizer of (24) for $T < \infty$ follows from the standard existence theorem for PDE-constrained optimization problems ([Hinze et al., 2009](#), Thm. 1.45).

Part II. [Lemma 2](#) implies first that the solution ρ of (19) is such that the function $\phi(t) = \frac{\rho(t)}{\rho_\infty} \xrightarrow{t \rightarrow \infty} C$ converges to a constant C . But we also have $\|\rho(t)\|_{L^1(\Omega)} = \|\rho_\infty(t)\|_{L^1(\Omega)} = 1$, since ρ and ρ_∞ are both probability density functions. Consequently, this constant is given by $C = 1$ and $\rho(t)$ converges to ρ_∞ in $L^2(\rho_\infty^{-1})$ for $t \rightarrow \infty$. By substituting $v = \mu_\infty \rho$ in (24), it follows that there exists for every $\delta > 0$ a $T_\delta < \infty$, with $T_\delta \rightarrow \infty$ for $\delta \rightarrow 0$, such that

$$\forall T \geq T_\delta, \quad \frac{J(T, \rho_0)}{T} \leq \ell_\infty + \delta. \quad (27)$$

Next, let (ρ_δ, v_δ) denote the minimizer of (24) on the time horizon $\mathbb{T} = (0, T_\delta)$ and let

$$\bar{\rho}_\delta \stackrel{\text{def}}{=} \frac{1}{T_\delta} \int_0^{T_\delta} \rho_\delta dt \quad \text{and} \quad \bar{v}_\delta \stackrel{\text{def}}{=} \frac{1}{T_\delta} \int_0^{T_\delta} v_\delta dt$$

denote their time averages. Since $\mathcal{K}$ is a convex cone, it follows that $(\bar{\rho}_\delta, \bar{v}_\delta) \in \mathcal{K}$ and

$$\mathcal{A}\bar{\rho}_\delta + \mathcal{B}\bar{v}_\delta = \frac{1}{T_\delta} \int_0^{T_\delta} \dot{\rho}_\delta dt = \frac{\rho(T_\delta) - \rho_0}{T_\delta},$$

which implies that

$$\lim_{\delta \rightarrow 0} (\mathcal{A}\bar{\rho}_\delta + \mathcal{B}\bar{v}_\delta) = 0, \quad (28)$$

almost everywhere on Ω , since ℓ is bounded from below and radially unbounded. In other words, $(\bar{\rho}_\delta, \bar{v}_\delta)$ converges to a feasible steady-state in $\mathcal{K}$. Moreover, Jensen's inequality yields

$$\frac{J(T_\delta, \rho)}{T_\delta} = \frac{\int_0^{T_\delta} \mathcal{L}(\rho_\delta, v_\delta) dt}{T_\delta} \geq \mathcal{L}(\bar{\rho}_\delta, \bar{v}_\delta). \quad (29)$$

But then (27) implies $\limsup_{\delta \rightarrow 0} \frac{J(T_\delta, \rho_0)}{T_\delta} \leq \ell_\infty$ and (28) and (29) and the fact that $\mathcal{A}$ is uniformly elliptic imply, by continuity, $\liminf_{\delta \rightarrow 0} \frac{J(T_\delta, \rho_0)}{T_\delta} \geq \ell_\infty$. Consequently, the ergodic limit for $\delta \rightarrow 0$, or, equivalently, $T_\delta \rightarrow \infty$, exists, and $J^* = \ell_\infty$. $\square$

4. Hamilton–Jacobi–Bellman equations

Many classical methods for analyzing solutions to HJBs ([Bardi & Capuzzo-Dolcetta, 1997](#); [Fleming & Soner, 1993](#)) start by establishing (or sometimes even imposing) certain properties – for example, Hölder continuity or differentiability – of the value function of an optimal control problem and then show that this function actually satisfies its associated HJB, be it in a classical, weak, or viscosity sense. An impressive exception is the work by Krylov ([Krylov, 1987, 2008](#)), where methods are developed for general nonlinear parabolic PDEs that can be applied to analyze the solutions of controlled diffusions and nonlinear HJBs. Nevertheless, it appears that the convex duality of HJBs and FPKs, which, in principle, is known since a long time ([Bismut, 1973](#); [Fleming & Vermes, 1989](#)), has never been fully exploited for the purpose of analyzing HJBs. Therefore, this section proposes to analyze HJBs by using modern PDE-constrained convex optimization methods ([Hinze et al., 2009](#)).

4.1. Hamiltonians

Let $H : \overline{\Omega} \times \mathbb{R}^{n_x} \rightarrow \mathbb{R}$ and $u^* : \overline{\Omega} \times \mathbb{R}^{n_x} \rightarrow \mathbb{U}$ denote the Hamiltonian and its associated minimizer,

$$H(x, \lambda) \stackrel{\text{def}}{=} \min_{u \in \mathbb{U}} \{ \ell(x, u) + \lambda^\top [f(x) + G(x)u] \}$$

$$\text{and } u^*(x, \lambda) \stackrel{\text{def}}{=} \operatorname{argmin}_{u \in \mathbb{U}} \{ \ell(x, u) + \lambda^\top [f(x) + G(x)u] \}$$

for all $(x, \lambda) \in \overline{\Omega} \times \mathbb{R}^{n_x}$. [Assumption 1](#) implies that u^* is Lipschitz on $\overline{\Omega} \times \mathbb{R}^{n_x}$. Thus, H is Lipschitz and

$$\forall V \in \mathcal{P}(\rho_\infty), \quad H(\cdot, \nabla V) \in L^2(0, T; L^2(\rho_\infty)). \quad (30)$$

The optimal steady state ρ_∞ is defined as in [Corollary 1](#).

4.2. Parabolic HJBs

On finite time horizons $\mathbb{T} = (0, T)$ the parabolic HJB with Neumann boundary condition is given by

$$\begin{aligned} -\dot{V} &= H(x, \nabla V) + \epsilon \Delta V && \text{on } \Omega_{\mathbb{T}} \\ 0 &= V(T) && \text{on } \Omega \\ 0 &= \nabla V^\top \eta && \text{on } \Sigma_{\mathbb{T}}. \end{aligned} \quad (31)$$

The definition of weak solutions of (31) relies on (30).

Definition 2. A function $V \in \mathcal{P}(\rho_\infty)$ with $V(T, \cdot) = 0$ is called a weak solution ([Hinze et al., 2009](#), Def. 1.27) of (31) if

$$-\int_0^T \langle \dot{V}, \rho \rangle dt = \int_0^T \int_\Omega [\rho H(x, \nabla V) - \epsilon \nabla \rho^\top \nabla V] dx dt$$

for all test functions $\rho \in \mathcal{P}(\rho_\infty^{-1})$.

Different from classical methods for HJBs, the following lemma uses Riesz' representation theorem to establish the existence and uniqueness of weak solutions in $\mathcal{P}(\rho_\infty)$.

Lemma 3. Let [Assumptions 1, 2, and 3](#) hold and let ρ_∞ be defined as in [Corollary 1](#). The statements below hold for all $T < \infty$.

1. HJB (31) has a unique weak solution $V \in \mathcal{P}(\rho_\infty)$.
2. The value function $J(T, \cdot)$ of (24) is a bounded linear operator, whose Riesz representation is given by

$$\forall \rho_0 \in \mathcal{P}_0(\rho_\infty^{-1}), \quad J(T, \rho_0) = \int_\Omega V(0, x) \rho_0(x) dx.$$

Proof. Let us temporarily assume that Ω is bounded. In this case, the auxiliary optimization problem

$$\begin{aligned} \mathfrak{J}(\mu) &\stackrel{\text{def}}{=} \min_{(\rho, v) \in \mathcal{K}_{\mathbb{T}}} \int_0^T \mathcal{L}(\rho, v) dt \\ \text{s.t. } &\begin{cases} \dot{\rho} &= \mathcal{A}\rho + \mathcal{B}v && \text{on } \Omega_{\mathbb{T}} \\ v &= \mu\rho && \text{on } \Omega_{\mathbb{T}} \\ \rho_0 &= \rho(0) && \text{on } \Omega \\ 0 &= (\epsilon \nabla \rho - f\rho - Gv)^\top \eta && \text{on } \Sigma_{\mathbb{T}} \end{cases} \end{aligned} \quad (32)$$

satisfies Slater's weak constraint qualification for uniformly parabolic PDE-constrained convex optimization problems on a bounded domain, recalling that $\mathcal{L}$ is strongly convex in v . Thus, it follows from a standard existence result for PDE-constrained optimization problems ([Hinze et al., 2009](#), Chapter 1.5) that (32) has for all parameters $\mu \in \mathbb{U}$ and given $\rho_0 \in \mathcal{P}_0(\rho_\infty^{-1})$ a unique minimizer $(\rho_\mu, v_\mu) \in \mathcal{K}_{\mathbb{T}}$. Next, let $V \in \mathcal{P}(\rho_\infty)$ denote the unique weak solution of the uniformly parabolic linear PDE

$$\begin{aligned} -\dot{V} &= \ell[\mu^*] + F[\mu^*]^\top \nabla V + \epsilon \Delta V && \text{on } \Omega_{\mathbb{T}} \\ 0 &= V(T) && \text{on } \Omega \\ 0 &= \nabla V^\top \eta && \text{on } \Sigma_{\mathbb{T}} \end{aligned} \quad (33)$$

at $\mu^* = v^*/\rho^*$, where $(\rho^*, v^*) \in \mathcal{K}_{\mathbb{T}}$ denotes – as in [Theorem 1](#) – the minimizer of (24). This notation is such that the relations $\rho^* = \rho_{\mu^*}$ and $v^* = v_{\mu^*}$ hold. Let

$$\chi(\rho, \mu) \stackrel{\text{def}}{=} \int_0^T \{ \langle \dot{V} + \ell[\mu] + F[\mu]^T \nabla V, \rho \rangle - \epsilon \nabla \rho^T \nabla V \} dt + \int_{\Omega} V(0, x) \rho_0(x) dx$$

denote an associated parametric Lagrangian at V . Because V is a weak solution of (33), the function $\chi(\cdot, \mu^*)$ is constant on $\mathcal{P}(\rho_{\infty}^{-1})$ and satisfies

$$\forall \rho \in \mathcal{P}(\rho_{\infty}^{-1}), \quad \chi(\rho, \mu^*) = \int_{\Omega} V(0, x) \rho_0(x) dx. \quad (34)$$

Next, we define $\hat{\mu} \stackrel{\text{def}}{=} u^*(\cdot, \nabla V)$, recalling that u^* is Lipschitz continuous. Hence, $\hat{\mu} \in \mathfrak{U}$ and

$$\begin{aligned} \inf_{\mu \in \mathfrak{U}} \chi(\rho, \mu) &= \chi(\rho, \hat{\mu}) \\ &= \int_0^T \int_{\Omega} \{ \rho H(x, \nabla V) - \epsilon \nabla \rho^T \nabla V \} dx dt \\ &\quad + \int_0^T \langle \dot{V}, \rho \rangle dt + \int_{\Omega} V(0, x) \rho_0(x) dx \end{aligned} \quad (35)$$

for all $\rho \in \mathcal{P}(\rho_{\infty}^{-1})$ with $\rho \geq 0$. Moreover, by regarding V as a test function, it follows that

$$\begin{aligned} \mathfrak{J}(\mu) &= \int_0^T \{ \mathfrak{L}(\rho_{\mu}, \mu \rho_{\mu}) - \langle \dot{\rho}_{\mu}, V \rangle \\ &\quad + a(\rho_{\mu}, V) + b(\mu \rho_{\mu}, V) \} dt \\ &= \chi(\rho_{\mu}, \mu), \end{aligned} \quad (36)$$

for all $\mu \in \mathfrak{U}$, where the second equation in (36) follows by substituting the definitions of a, b, χ , and $\mathfrak{L}$ and a partial integration with respect to time. Now, we have

$$\begin{aligned} \mathfrak{J}(\mu^*) &\stackrel{(36)}{=} \chi(\rho^*, \mu^*) \stackrel{(34)}{=} \chi(\rho_{\hat{\mu}}, \mu^*) \geq \inf_{\mu \in \mathfrak{U}} \chi(\rho_{\hat{\mu}}, \mu) \\ &\stackrel{(35)}{=} \chi(\rho_{\hat{\mu}}, \hat{\mu}) \stackrel{(36)}{=} \mathfrak{J}(\hat{\mu}). \end{aligned} \quad (37)$$

Since $\mu^* = v^*/\rho^*$ is – due to [Theorem 1](#) – the unique minimizer of the optimization problem

$$\mathfrak{J}(\mu^*) = \min_{\mu \in \mathfrak{U}} \mathfrak{J}(\mu) \leq \mathfrak{J}(\hat{\mu}) \stackrel{(37)}{\implies} \mathfrak{J}(\mu^*) = \mathfrak{J}(\hat{\mu}),$$

it follows from (37) that $\hat{\mu} = \mu^*$. Moreover, by substituting the equation $\mu^* = \hat{\mu}$ in (33), we find that V satisfies (31); and, finally, (34) and (36) yield

$$J(T, \rho_0) = \mathfrak{J}(\mu^*) = \int_{\Omega} V(0, x) \rho_0(x) dx$$

for all $\rho_0 \in \mathcal{P}(\rho_{\infty}^{-1})$. This implies that $J(T, \cdot)$ is a bounded linear operator. Next, if HJB (31) had a second solution $\tilde{V} \neq V$, we could repeat the above construction by using the control law $\tilde{\mu} = u^*(\cdot, \nabla \tilde{V})$ in order to find yet another representation of $J(T, \cdot)$, namely,

$$J(T, \rho_0) = \int_{\Omega} \tilde{V}(0, x) \rho_0(x) dx$$

for any $\rho_0 \in \mathcal{P}(\rho_{\infty}^{-1})$. But this is impossible, as the Riesz representation of the bounded linear operator $J(T, \cdot)$ must be unique. Thus, HJB (31) has a unique solution and we have proven both statements of the theorem under the additional assumption that Ω is bounded.

Last but not least, if Ω is unbounded, the same strategy as in the proof of [Lemma 2](#) can be applied. This means that we

repeat the above argument by replacing the domain Ω in (24) with an increasing sequence of bounded domains Ω_{α_k} in order to construct an associated sequence of bounded linear operators, $J_k(T, \cdot)$, that converges to $J(T, \cdot)$ for $k \rightarrow \infty$. Since [Theorem 1](#) has already established the fact that $J(T, \cdot)$ is bounded for finite T but potentially unbounded domains Ω , it follows that $J(T, \cdot)$ is a bounded linear operator. Note that this is equivalent to the convergence of the corresponding sequence of Riesz' representations $V_k \in \mathcal{P}(\rho_{\infty})$ of

$$J_k(T, \rho_0) = \int_{\Omega_{\alpha_k}} V_k(0, x) \rho_0(x) dx,$$

since in Hilbert spaces one can rely on the natural isometric isomorphism between linear operators and their Riesz' representations. Because H is Lipschitz continuous, the limit of the sequence V_k for $k \rightarrow \infty$ is the unique weak solution of (31). This completes our proof. $\square$

Remark 1. As mentioned above, parabolic HJBs are among the most well-studied nonlinear PDEs in the literature ([Bardi & Capuzzo-Dolcetta, 1997](#); [Fleming & Soner, 1993](#); [Krylov, 2008](#)). [Lemma 3](#) adds yet another set of assumptions under which such HJBs admit a unique solution. The relevance of this result, however, is that – different from all other methods for analyzing the existence of solutions of HJBs – the proof of [Lemma 3](#) starts from the convex optimization problem (24) and then derives the HJB (31) by constructing an associated dual problem.

5. Exponential stabilizability

Carré Du Champ (CDC) operators have originally been introduced in the French literature on functional analysis ([Bakry & Émery, 1985](#)). The following section introduces a variant of such CDC operators in order to establish global exponential stabilizability conditions for controlled FPKs.

5.1. Weighted Carré Du Champ operators

The weighted CDC operator Γ is a bilinear form,

$$\forall \varphi, \phi \in C^\infty(\Omega), \quad \Gamma(\varphi, \phi) \stackrel{\text{def}}{=} \nabla \varphi^T P \nabla \phi. \quad (38)$$

Here, $P \in C^2(\Omega)^{n_x \times n_x}$ denotes a weight that satisfies $\underline{\lambda} I \preceq P \preceq \bar{\lambda} I$ for given positive lower- and upper bounds, $0 < \underline{\lambda} \leq \bar{\lambda} < \infty$. If $P = \epsilon I$, Γ coincides with the traditional CDC operator of the diffusion semigroup that is generated by the linear operator

$$\forall \phi \in C^\infty(\Omega), \quad \mathcal{L}[\mu]\phi \stackrel{\text{def}}{=} F[\mu]^T \nabla \phi + \epsilon \Delta \phi.$$

It is defined for all control laws $\mu \in \mathfrak{U}$, although the right-hand of the CDC inequality

$$\frac{1}{2} (\mathcal{L}[\mu]\phi^2 - 2\phi \mathcal{L}[\mu]\phi) = \epsilon \|\nabla \phi\|_2^2 \leq \frac{\epsilon}{\underline{\lambda}} \Gamma(\phi, \phi) \quad (39)$$

is an invariant, in the sense that it does not depend on μ .

5.2. Lyapunov diffusion operators

The design of potentially non-trivial weight functions P turns out to be important for the construction of practical stabilizability conditions. In order to explain why this is so, the Lyapunov diffusion operator

$$\mathcal{R}[\mu]P \stackrel{\text{def}}{=} \frac{1}{2} (\mathcal{L}[\mu]P - F'[\mu]P - PF'[\mu]^T), \quad (40)$$

is introduced for arbitrary $\mu \in \mathcal{U}$. Here, $\mathcal{L}[\mu]$ is applied to $P \in C^2(\Omega)^{n_x \times n_x}$ componentwise and $F'[\mu]$ denotes the distributional Jacobian of $F[\mu]$. In detail, we define

$$(\mathcal{L}[\mu]P)_{i,j} \stackrel{\text{def}}{=} F[\mu]^\top \nabla P_{i,j} + \epsilon \Delta P_{i,j}$$

$$\text{and } (F'[\mu])_{i,j} \stackrel{\text{def}}{=} \partial_{x_j} F_i[\mu]$$

for all components $i, j \in \{1, \dots, n_x\}$, where ∂_{x_j} denotes the distributional derivative with respect to x_j . Here, we say that a symmetric matrix-valued distribution Y on Ω is positive semi-definite, writing $Y \geq 0$, if $\int_\Omega \text{Tr}(YZ) dx \geq 0$ for all symmetric and positive semi-definite test functions $Z \in C_0^\infty(\Omega)^{n_y \times n_y}$ with compatible dimension. Linear matrix inequalities (LMIs) are then understood in an analogous manner.

Assumption 4. Let $\mu_\infty = \nu_\infty/\rho_\infty$ denote the optimal ergodic control law as defined in Corollary 1. We assume that there exists a $\lambda > 0$ such that the LMI

$$\begin{pmatrix} \mathcal{R}[\mu_\infty]P & \nabla^\top \otimes (\epsilon P) \\ \nabla \otimes (\epsilon P) & I \otimes (\epsilon P) \end{pmatrix} \geq \lambda \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} \text{ on } \Omega \quad (41)$$

admits a solution $P \in C^2(\Omega)^{n_x \times n_x}$ with $\underline{\lambda}I \leq P \leq \bar{\lambda}P$ on Ω . Here, $\otimes$ denotes the Kronecker product.

Example 3. Let $f(x) = Ax$ and $G(x) = B$ be a linear control system on $\Omega = \mathbb{R}^{n_x}$ with (A, B) being asymptotically stabilizable and $\ell(x, u) = \|x\|^2 + \|u\|^2$. In this case, $\mu_\infty(x) = Kx$ is the well-known Linear-Quadratic-Gaussian (LQG) regulator and Assumption 4 holds for a constant $P \in \mathbb{R}^{n_x \times n_x}$. This is because for a constant $P > 0$, the equations $\mathcal{L}[\mu]P = 0$ and $\nabla \otimes P = 0$ hold such that LMI (41) reduces to

$$\mathcal{R}[\mu]P = -\frac{(A+BK)P + P(A+BK)^\top}{2} \geq \lambda P.$$

This LMI has a strictly positive solution $P > 0$ for sufficiently small $\lambda > 0$, since we assume that (A, B) is asymptotically stabilizable, or, equivalently, that the eigenvalues of $A+BK$ have strictly negative real parts.

Example 4. Let us consider a stochastic system with $f(x) = \frac{x}{2} - x^3$, $G(x) = 0$, and $\epsilon = 1$ on $\Omega = \mathbb{R}$. It leads to an SDE with multi-modal limit distribution,

$$\rho_\infty(x) = \frac{1}{C} \exp\left(\frac{x^2 - x^4}{4}\right),$$

where the constant C can be chosen such that the integral over ρ_∞ is equal to 1. In this example, (41) cannot be satisfied for a constant function P , but it is possible to find other classes of functions which satisfy (41). For instance, Assumption 4 can be satisfied by setting

$$P(x) = 1 - \frac{1}{2}e^{-x^2}, \quad \lambda = \frac{1}{20}, \quad \underline{\lambda} = \frac{1}{2}, \quad \text{and } \bar{\lambda} = 1.$$

Apart from the above examples, another reason why Assumption 4 appears appropriate for our purposes, is elaborated in the following section. Namely, it turns out that LMI (41) is satisfied for an even broader class of problems than the original Bakry-Emery condition for FPKs, which is – at least in the FPK literature – well-accepted as a useful and practical condition; see, for instance, (Arnold et al., 2001; Bakry & Émery, 1985; Bolley & Gentil, 2010; Ledoux, 1992), where many examples are collected.

5.3. Exponential convergence rate

The following lemma interprets LMI (41) as a generalized Bakry-Emery condition.

Lemma 4. Let Assumptions 1, 2, and 4 hold and let $\rho_0 \in \mathcal{P}_0(\rho_\infty^{-1})$ be given. Then the controlled FPK

$$\begin{aligned} \dot{\rho} &= \mathcal{A}\rho + \mathcal{B}(\mu_\infty\rho) && \text{on } \Omega_\mathbb{T} \\ \rho_0 &= \rho(0) && \text{on } \Omega \\ 0 &= (\nabla\rho - F[\mu_\infty]\rho)^\top \eta|_{\partial\Omega} && \text{on } \Sigma_\mathbb{T} \end{aligned} \quad (42)$$

has a unique solution $\rho \in \mathcal{P}(\rho_\infty^{-1})$ that converges to the optimal ergodic density function ρ_∞ in $L^2(\rho_\infty^{-1})$ with exponential convergence rate $\gamma = 2\lambda \cdot (\underline{\lambda}/\bar{\lambda})$; that is,

$$\forall t > 0, \quad \|\rho(t) - \rho_\infty\|_{L^2(\rho_\infty^{-1})} \leq e^{-\gamma t} \|\rho_0 - \rho_\infty\|_{L^2(\rho_\infty^{-1})}. \quad (43)$$

Proof. In order to interpret LMI (41) as a generalized Bakry-Emery condition, we work out the Γ_2 -operator

$$\begin{aligned} \Gamma_2(\phi) &\stackrel{\text{def}}{=} \frac{1}{2} (\mathcal{L}[\mu_\infty]\Gamma(\phi, \phi) - 2\Gamma(\phi, \mathcal{L}[\mu_\infty]\phi)) \\ &= \frac{1}{2} F[\mu_\infty]^\top \nabla (\nabla\phi^\top P \nabla\phi) + \epsilon \Delta (\nabla\phi^\top P \nabla\phi) \\ &\quad - \nabla\phi P \nabla (F[\mu_\infty]^\top \nabla\phi + \epsilon \Delta\phi) \\ &= \frac{1}{2} \nabla\phi^\top \mathcal{L}[\mu_\infty] P \nabla\phi + 2\epsilon (\nabla \otimes \nabla\phi)^\top (\nabla \otimes P) \nabla\phi \\ &\quad + \epsilon (\nabla \otimes \nabla\phi)^\top (I \otimes P) (\nabla \otimes \nabla\phi) \\ &\quad - \nabla\phi P F'[\mu_\infty]^\top \nabla\phi \\ &= \begin{pmatrix} \nabla\phi \\ \nabla \otimes \nabla\phi \end{pmatrix}^\top \begin{pmatrix} \mathcal{R}[\mu_\infty]P & \epsilon \nabla^\top \otimes P \\ \epsilon \nabla \otimes P & \epsilon I \otimes P \end{pmatrix} \begin{pmatrix} \nabla\phi \\ \nabla \otimes \nabla\phi \end{pmatrix} \end{aligned}$$

for arbitrary test functions $\phi \in C_0^\infty(\bar{\Omega})$. Note that this equation is understood by interpreting all first order derivatives of $F[\mu_\infty]$ as distributional derivatives. Next, (41) implies that the integral version of the Bakry-Emery condition (Bakry & Émery, 1985; Ledoux, 1992),

$$\int_\Omega \Gamma_2(\phi)\omega dx \geq \lambda \int_\Omega \Gamma(\phi, \phi)\omega dx, \quad (44)$$

is satisfied for all test functions $\phi \in C_0^\infty(\bar{\Omega})$ and all weight functions $\omega \in H^1(\Omega)$ with $\omega \geq 0$. This integral inequality remains correct for distributional Jacobians $F'[\mu_\infty]$, since these distributional derivatives merely enter via the integrand of the left-hand integral in (44), and, consequently, can be resolved by partial integration. It is well-known (Bolley & Gentil, 2010, Thm. 2 & Prop. 5) that (39) and (44) imply that the entropy inequality

$$\int_\Omega (\phi - 1)^2 \rho_\infty dx \leq \frac{\epsilon}{2\lambda\underline{\lambda}} \int_\Omega \Gamma(\phi, \phi) \rho_\infty dx, \quad (45)$$

holds for all $\phi \in C_0^\infty(\bar{\Omega})$, recalling that $(\rho_\infty dx)$ is, by construction, an ergodic measure of the diffusion operator $\mathcal{L}[\mu_\infty]$. Of course, by continuity, it follows that (45) then also holds for all functions $\phi \in H^1(\rho_\infty)$. Consequently, it further follows from Lemma 2 that

$$\begin{aligned} \frac{d}{dt} \|\rho(t) - \rho_\infty\|_{L^2(\rho_\infty^{-1})}^2 &\leq \frac{-2\epsilon}{\bar{\lambda}} \int_\Omega \Gamma(\phi, \phi) \rho_\infty dx \\ &\stackrel{(45)}{\leq} -4\lambda(\underline{\lambda}/\bar{\lambda}) \int_\Omega (\phi - 1)^2 \rho_\infty dx \\ &= -2\gamma \|\rho(t) - \rho_\infty\|_{L^2(\rho_\infty^{-1})}^2, \end{aligned} \quad (46)$$

where we have set $\phi = \rho/\rho_\infty$. An application of Gronwall's lemma to (46) yields the energy estimate (43), which implies that the unique weak solution $\rho \in \mathcal{P}(\rho_\infty^{-1})$ of (42) remains well-defined on infinite horizons. $\square$

6. Infinite horizon optimal control

The result of Lemma 4 on the exponential convergence of the state of the optimally controlled FPK can be used to bound the value function of infinite horizon optimal control problems. Here, the main idea is to exploit the strong duality between FPKs and HJBs from Lemma 3.

6.1. Problem formulation

This section concerns infinite horizon optimal control problems of the form

$$J_\infty(\rho_0) \stackrel{\text{def}}{=} \lim_{T \rightarrow \infty} (J(T, \rho_0) - T\ell_\infty), \quad (47)$$

recalling that the cost function J has been defined in (24). Here, ℓ_∞ denotes the ergodic average cost that has been defined in (18) and interpreted in Theorem 1. Compared to the ergodic optimal control problem from Theorem 1, the cost J_∞ of the infinite horizon optimal control problem (47) is more difficult to analyze, as it depends on the initial probability density function $\rho_0 \in \mathcal{P}_0(\rho_\infty)$. Conditions under which the limit on the right hand of (47) exists are discussed below.

6.2. Ergodic HJB

Problem (47) is closely related to the ergodic HJB with Neumann boundary condition, given by

$$\begin{aligned} \ell_\infty &= H(x, \nabla V_\infty) + \epsilon \Delta V_\infty & \text{on } \Omega \\ 0 &= (\nabla V_\infty)^\top \eta & \text{on } \partial\Omega. \end{aligned} \quad (48)$$

Definition 3. A function $V_\infty \in H^1(\rho_\infty)$ is called a weak solution (Hinze et al., 2009, Def. 1.20) of (48) if

$$\ell_\infty \int_\Omega \rho \, dx = \int_\Omega [\rho H(x, \nabla V_\infty) - \epsilon \nabla \rho^\top \nabla V_\infty] \, dx$$

for all test functions $\rho \in H^1(\rho_\infty^{-1})$.

The relation between this ergodic HJB and (47) is explained in the following section.

6.3. Solution of the infinite horizon control problem

The following theorem summarizes a novel result about the existence of weak solutions of ergodic HJBs in the presence of general indefinite stage cost functions.

Theorem 2. If Assumptions 1, 2, 3, and 4 hold,¹ then the limit J_∞ in (47) exists for all $\rho_0 \in \mathcal{P}_0(\rho_\infty^{-1})$ and there exists a unique $V_\infty \in H^1(\rho_\infty)$ such that

$$\forall \rho_0 \in \mathcal{P}_0(\rho_\infty^{-1}), \quad J_\infty(\rho_0) = \int_\Omega V_\infty(x) \rho_0(x) \, dx.$$

Moreover, V_∞ is a weak solution of the ergodic HJB (48). It satisfies the energy estimate

$$\int_\Omega (V_\infty)^2 \rho_\infty \, dx \leq \frac{1}{\gamma^2} \int_\Omega \ell[\mu_\infty]^2 \rho_\infty \, dx, \quad (49)$$

recalling that $\gamma = 2\lambda \cdot (\underline{\lambda}/\bar{\lambda})$.

Proof. The proof is divided into three parts:

Part I. Let ρ denote the solution of (42) and let $v = \mu_\infty \rho$ denote its associated input function such that

$$\begin{aligned} |\mathfrak{L}(\rho, v) - \ell_\infty| &\stackrel{(26)}{\leq} \|\rho - \rho_\infty\|_{L^2(\rho_\infty^{-1})} \|\ell[\mu_\infty]\|_{L^2(\rho_\infty)} \\ &\stackrel{(43)}{\leq} e^{-\gamma t} \|\rho_0 - \rho_\infty\|_{L^2(\rho_\infty^{-1})} \|\ell[\mu_\infty]\|_{L^2(\rho_\infty)}. \end{aligned}$$

¹ The proof of Theorem 2 relies on the exponential convergence rate estimate (43) which holds if the Bakry–Emery type condition from Assumption 4 is met. As there may exist other variants and generalizations of the CDC condition, see also Bakry and Émery (1985) and Bolley and Gentil (2010), we emphasize that the statement of Theorem 2 remains valid under any such conditions, as long as they guarantee that (43) holds for some constant $\gamma > 0$.

The above upper bounded can be used to bound the value function of the auxiliary optimization problem

$$\begin{aligned} J^+(T, \rho_0) &\stackrel{\text{def}}{=} \min_{\rho, v} \int_0^T (\mathfrak{L}(\rho, v) - \ell_\infty) \, dt \\ \text{s.t. } \begin{cases} \dot{\rho} &= \mathcal{A}\rho + \mathcal{B}v & \text{on } \Omega_T \\ v &= \mu_\infty \rho & \text{on } \Omega_T \\ \rho_0 &= \rho(0) & \text{on } \Omega \\ 0 &= (\epsilon \nabla \rho - f\rho - Gv)^\top \eta & \text{on } \Sigma_T. \end{cases} \end{aligned} \quad (50)$$

Namely, by integrating the above inequality, we find that

$$|J^+(T, \rho_0)| \leq \frac{\|\rho_0 - \rho_\infty\|_{L^2(\rho_\infty^{-1})}}{\gamma} \sqrt{\int_\Omega \ell[\mu_\infty]^2 \rho_\infty \, dx} \quad (51)$$

for all $T > 0$ and all $\rho_0 \in \mathcal{P}_0(\rho_\infty^{-1})$. As in the proof of Lemma 3, we can use the weak solution of

$$\begin{aligned} -\dot{V}^+ &= \ell[\mu_\infty] - \ell_\infty + F[\mu_\infty]^\top \nabla V^+ + \epsilon \Delta V^+ & \text{on } \Omega_T \\ 0 &= V^+(T) & \text{on } \Omega \\ 0 &= \nabla(V^+)^\top \eta & \text{on } \Sigma_T \end{aligned}$$

to represent $J^+(T, \rho)$ in the form

$$\forall \rho_0 \in \mathcal{P}_0(\rho_\infty^{-1}), \quad J^+(T, \rho_0) = \int_\Omega V^+(0, x) \rho_0(x) \, dx.$$

Next, because $J(T, \cdot)$ is uniformly bounded, its limit for $T \rightarrow \infty$ admits a unique Riesz representation, $J_\infty^+(\rho_0) \stackrel{\text{def}}{=} \lim_{T \rightarrow \infty} J^+(T, \rho_0) = \int_\Omega V_\infty(x) \rho_0(x) \, dx$, where $V_\infty \in H^1(\rho_\infty)$ must be a weak solution of

$$\begin{aligned} \ell_\infty &= \ell[\mu_\infty] + F[\mu_\infty]^\top \nabla V_\infty + \epsilon \Delta V_\infty & \text{on } \Omega \\ 0 &= \nabla(V_\infty)^\top \eta & \text{on } \partial\Omega \end{aligned} \quad (52)$$

at $\mu_\infty = v_\infty/\rho_\infty$.

Part II. Let us use the function J_∞^+ as a terminal cost of the finite horizon optimal control problem

$$\begin{aligned} \tilde{J}(T, \rho_0) &\stackrel{\text{def}}{=} \min_{(\rho, v) \in \mathcal{K}_T} \int_0^T (\mathfrak{L}(\rho, v) - \ell_\infty) \, dt + J_\infty^+(\rho(T)) \\ \text{s.t. } \begin{cases} \dot{\rho} &= \mathcal{A}\rho + \mathcal{B}v & \text{on } \Omega_T \\ \rho_0 &= \rho(0) & \text{on } \Omega \\ 0 &= (\epsilon \nabla \rho - f\rho - Gv)^\top \eta & \text{on } \Sigma_T. \end{cases} \end{aligned} \quad (53)$$

By using an analogous argument as in Lemma 3, it follows that $\tilde{J}(T, \cdot)$ is a bounded linear operator, whose unique Riesz representation $\tilde{J}(T, \rho_0) = \int_\Omega \tilde{V}(0, x) \rho_0(x) \, dx$ is given by the weak solution of the parabolic HJB

$$\begin{aligned} -\partial_t \tilde{V} &= H(x, \nabla \tilde{V}) + \epsilon \Delta \tilde{V} & \text{on } \Omega_T \\ V_\infty &= \tilde{V}(T) & \text{on } \Omega \\ 0 &= \nabla \tilde{V}^\top \eta & \text{on } \Sigma_T. \end{aligned} \quad (54)$$

Since μ_∞ is a feasible time-invariant control law for (53), we must have

$$\forall x \in \Omega, \quad \tilde{V}(0, x) \leq V_\infty(x).$$

Next, let us assume that we can find a $\rho_0 \in \mathcal{P}_0(\rho_\infty^{-1})$ and a $\tilde{T} > 0$ for which $\tilde{J}(\tilde{T}, \rho_0) < J_\infty^+(\rho_0)$. Then the Lebesgue measure of the set $\{x \in \Omega \mid \tilde{V}(0, x) < V_\infty(x)\}$ must be strictly positive. But this implies that there exists a $\delta > 0$ such that

$$\forall T \geq \tilde{T}, \quad \tilde{J}(T, \rho_\infty) \leq J_\infty^+(\rho_\infty) - \delta = -\delta, \quad (55)$$

where δ does not depend on T . In order to show that this inequality leads to a contradiction, we denote the minimizer of (53) at $\rho_0 = \rho_\infty$ by (ρ_T^*, v_T^*) . Due to the above inequality, the ergodic average values

$$\bar{\rho} \stackrel{\text{def}}{=} \lim_{T \rightarrow \infty} \frac{\int_0^T \rho_T^* \, dt}{T} \quad \text{and} \quad \bar{v} \stackrel{\text{def}}{=} \lim_{T \rightarrow \infty} \frac{\int_0^T v_T^* \, dt}{T}$$

exist in $H^1(\Omega)$, since the optimal trajectories remain bounded and must satisfy $\lim_{T \rightarrow \infty} \rho_T^*(T) = \rho_\infty$ and $\lim_{T \rightarrow \infty} v_T^*(T) = v_\infty$, where the limits are understood in $H^1(\Omega)$. The latter claim follows by using an analogous argument as in the proof of [Theorem 1](#), recalling that the ergodic limit (ρ_∞, v_∞) is strictly optimal on infinite horizons. This implies $0 = \mathcal{A}\bar{\rho} + \mathcal{B}\bar{v}$ and $\mathcal{L}(\bar{\rho}, \bar{v}) \leq \ell_\infty - \delta$, due to (55) and Jensen's inequality, recalling that $\mathcal{L}$ is convex. But this contradicts the definition of ℓ_∞ in (18). Thus,

$$\forall T > 0, \quad \tilde{J}(T, \cdot) = J_\infty^+(\cdot) = J_\infty(\cdot),$$

where the latter equation follows from the fact the first equation holds uniformly and the fact that the contribution of the terminal cost, $\lim_{T \rightarrow \infty} J_\infty^+(\rho_T^*(T)) = 0$ vanishes for $T \rightarrow \infty$.

Part III. An immediate consequence of the constructions in Part II is that the unique weak solution of the parabolic HJB (54),

$$\forall x \in \Omega, \quad \tilde{V}(t, x) = V_\infty(x)$$

does not depend on $t > 0$. Thus, V_∞ is a weak solution of the ergodic HJB (48). Finally, (51) follows from

$$\begin{aligned} \|V_\infty\|_{L^2(\rho_\infty)} &= \sup_{\rho_0 \neq \rho_\infty} \frac{\int_\Omega V_\infty(\rho_0 - \rho_\infty) dx}{\|\rho_0 - \rho_\infty\|_{L^2(\rho_\infty^{-1})}} \\ &= \sup_{\rho_0 \neq \rho_\infty} \frac{J_\infty^+(\rho_0)}{\|\rho_0 - \rho_\infty\|_{L^2(\rho_\infty^{-1})}} \\ &\stackrel{(51)}{\leq} \frac{1}{\gamma} \sqrt{\int_\Omega \ell[\mu_\infty]^2 \rho_\infty dx}. \end{aligned} \quad (56)$$

This completes the proof. $\square$

Remark 2. [Theorem 2](#) implies that the value function satisfies $V_\infty \in H^1(\omega)$ for the ergodic weight $\omega = \rho_\infty$. If Ω is bounded, this implies $V_\infty \in H^1(1)$, as ρ_∞ is strictly positive. If Ω is unbounded, however, the weight is important. For example, for the Ornstein–Uhlenberg process,

$$f(x) = -x, \quad g(x) = 1, \quad \ell(x, u) = 3x^2 + u^2, \quad \Omega = \mathbb{U} = \mathbb{R},$$

we have $V_\infty(x) = x^2$ and $\rho_\infty(x) = \frac{e^{-x^2/\epsilon}}{\sqrt{\pi\epsilon}}$. In this case, V_∞ is not square integrable, but we have $V_\infty \in H^1(\rho_\infty)$.

7. Global optimal control algorithm

The results of [Lemma 3](#) and [Theorem 2](#) motivate a completely new numerical method for solving HJBs. This method relies on discretizing the dynamic programming recursion for the bounded linear operator $J(T, \cdot)$ rather than on discretizing the value function V directly.

7.1. Galerkin projection

The bounded linear operator $J(T, \cdot)$, as defined in (24), can be approximated by a finite dimensional linear function $J_0(\cdot)$ by applying a simple Galerkin projection. This means that we choose basis functions $\varphi_1, \varphi_2, \dots, \varphi_m \in H^1(\Omega)$ in order to discretize the states ρ and controls v . For simplicity of presentation, we assume that this construction is based on given grid points $x_1, x_2, \dots, x_m \in \Omega$ and $t_k = kh$ with step-size $h > 0$ and time index $k \in \mathbb{N}$ and that Lagrange's equation holds²; that is, $\varphi_i(x_j) = \delta_{i,j}$, where $\delta_{i,j} = 0$ if $i \neq j$ and $\delta_{i,i} = 1$. In this case, the

² Note that the methods in this paper can be extended easily for all kinds of finite element-, model order reduction, and other related numerical methods for linear parabolic PDEs or even tailored methods for FPKs, which might use basis functions that do not satisfy Lagrange's equation—this equation is merely introduced in order avoid unnecessary technical notation overhead.

discrete approximation of the bounded linear operator $J(T, \cdot)$ at $T = t_N$ has the form

$$\begin{aligned} J_0(z_0) &\stackrel{\text{def}}{=} \min_{z, v} \sum_{k=0}^{N-1} \sum_{i=1}^m w_i \cdot \ell\left(x_i, \frac{v_{k,i}}{z_{k,i}}\right) \cdot z_{k,i} \\ \text{s.t.} \quad &\begin{cases} \forall k \in \{0, 1, \dots, N-1\}, \\ \mathbf{E}z_{k+1} = \mathbf{A}z_k + \mathbf{B}v_k \\ (z_{k,i}, v_{k,i}) \in \mathbb{K} \quad \text{f.a.} \quad i \in \{1, \dots, m\}. \end{cases} \end{aligned} \quad (57)$$

Here, $\mathbf{A}$ and $\mathbf{B}$ are discrete-time system matrices that are found by a Galerkin projection of $\mathcal{A}$ and $\mathcal{B}$ in combination with a suitable implicit differential equation solver in order to discretize the system also in time. In this context, $\mathbf{E}$ is an invertible mass matrix. The coefficients of the associated truncated series expansions $\rho(t_k) \approx \sum_{i=1}^m z_{k,i} \varphi_i$ and $v(t_k) \approx \sum_{i=1}^m v_{k,i} \varphi_i$ are optimized subject to a conic constraint, given by

$$\mathbb{K} = \left\{ (z, v) \in \mathbb{R} \times \mathbb{R}^{nu} \mid v \in z\mathbb{U} \text{ and } z > 0 \right\}.$$

The quadrature weights $w_i > 0$ are needed to discretize the integral in the objective of (24). Methods for bounding the discretization error of the Galerkin projection of (24) in dependence on m and N can be found in [Brenner and Scott \(2005\)](#).

7.2. Dual perspective function

For the following developments it will be convenient to introduce the Fenchel conjugate of the stage cost,

$$d(x, \lambda) \stackrel{\text{def}}{=} \min_{u \in \mathbb{U}} \{ \ell(x, u) + \lambda^\top u \}.$$

Its associated dual perspective function is defined by

$$D(\lambda) \stackrel{\text{def}}{=} \begin{pmatrix} d(x_1, \lambda_1/w_1) \cdot w_1 \\ \vdots \\ d(x_m, \lambda_m/w_m) \cdot w_m \end{pmatrix}.$$

7.3. Operator-theoretic dynamic programming

Assuming that the stochastic optimal control problem is discretized as above, it can be solved by starting with $y_N = 0$ and solving the recursion

$$\mathbf{E}^\top y_k = \mathbf{A}^\top y_{k+1} + \mathbf{D}(\mathbf{B}^\top y_{k+1}) \quad (58)$$

backwards in time, for $k \in \{N-1, \dots, 1, 0\}$. This procedure can be justified directly by observing that the linear cost-to-go function of the discrete optimal control problem (57) is given by $J_k(z) = y_k^\top \mathbf{E}z$. It satisfies the discretized operator-theoretic dynamic programming recursion,

$$\begin{aligned} y_k^\top \mathbf{E}z &= \min_v \sum_{i=1}^m w_i \cdot \ell\left(x_i, \frac{v_i}{z_i}\right) \cdot z_i + y_{k+1}^\top (\mathbf{A}z + \mathbf{B}v) \\ \text{s.t.} \quad &(z_i, v_i) \in \mathbb{K} \quad \text{f.a.} \quad i \in \{1, \dots, m\} \\ &= [\mathbf{A}^\top y_{k+1} + \mathbf{D}(\mathbf{B}^\top y_{k+1})]^\top z. \end{aligned}$$

A comparison of coefficients yields (58). Moreover, the associated separable minimizers

$$u_{k,i}^* \stackrel{\text{def}}{=} \argmin_{u \in \mathbb{U}} \left\{ \ell(x_i, u) + \frac{y_{k+1}^\top \mathbf{B}_i u}{w_i} \right\}$$

can be used to recover an approximation of the optimal control law $\mu^* = v^*/\rho^*$ at the minimizer (ρ^*, v^*) of (24), $\mu^*(t_k, x_i) \approx u_{k,i}^*$. The computational complexity for computing this approximation of μ^* is given by $\mathcal{O}(m \cdot N)$. Finally, regarding the solution of the infinite horizon control problem, one option is to keep on iterating (58) backward in time until a suitable termination criterion is satisfied, for example, until $\|u_k - u_{k+1}\|_\infty \leq \text{TOL}$

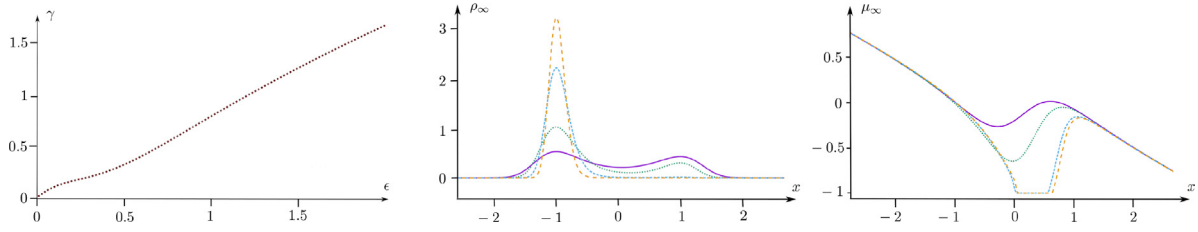


Fig. 1. LEFT: Exponential decay rate γ in dependence on ϵ . Since γ tends to 0 for $\epsilon \rightarrow 0^+$, the energy bound (49) diverges in the limit. In fact, for this example, the ergodic HJB has no viscosity solution for $\epsilon = 0$. MIDDLE: Optimal ergodic distribution ρ_∞ for $\epsilon = 1$ (solid line), $\epsilon = 0.5$ (dotted line), $\epsilon = 0.2$ (thin-dashed line), and $\epsilon = 0.1$ (wide-dashed line) RIGHT: the optimal control infinite horizon control law μ_∞ using the same values for ϵ .

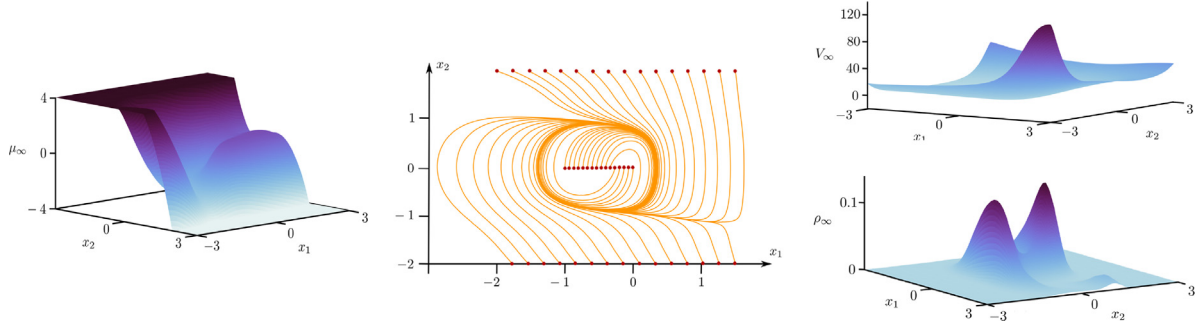


Fig. 2. LEFT: the optimal ergodic control law $\mu_\infty : \Omega \rightarrow \mathbb{U}$ on the domain $\Omega = [-3, 3]^2$ for the control bounds $\mathbb{U} = [-4, 4]$. MIDDLE: simulations of the states of the closed-loop system without noise for selected initial values (red dots), which converge to a non-trivial attractor set. UPPER RIGHT: the indefinite solution V_∞ of the ergodic HJB that corresponds to the unique Riesz representation of the cost function, as introduced in Theorem 2. LOWER RIGHT: the optimal multi-modal steady state ρ_∞ .

for a user specified tolerance $\text{TOL} > 0$. This method converges for $k \rightarrow -\infty$ as long as the conditions of Theorem 2 are satisfied—assuming that the above finite element discretization is reasonably accurate. Alternatively, one can use a large-scale convex optimization problem solver to solve (57) for $N = 1$ but subject to the additional stationarity constraint $z_1 = z_0$.

7.4. First case study

Let us consider a scalar system with $f(x) = 3(x - x^3)$, $g(x) = 1$, $\mathbb{U} = [-1, 1]$ and $\ell(x, u) = \frac{x}{10} + (x - 1)^2(x + 1)^2 + \frac{u^2}{2}$ on the domain $\Omega = [-3, 3]$. For $\epsilon = 0$, there exist two locally optimal steady states near the points -1 and 1 . However, due to the control bounds, transitioning from one steady-state to the other is infeasible. Thus, for $\epsilon = 0$, the ergodic Hamilton–Jacobi–Bellman (HJB) equation does not admit a global solution on the domain Ω . This stands in contrast to the scenario where $\epsilon > 0$, for which Theorems 1 and 2 guarantee the uniqueness and existence of solutions to both the ergodic and infinite horizon optimal control problem. The corresponding optimal ergodic steady-state ρ_∞ and control law μ_∞ are depicted in the middle and right panels of Fig. 1.

For $\epsilon = 1$ and $\epsilon = 0.5$, the optimal steady-state distribution exhibits a multi-modal characteristics, as anticipated. For smaller values of ϵ , however, ρ_∞ becomes centered around the globally optimal steady-state near -1 . This may initially appear counter-intuitive, given that transitioning to this point is impossible for $\epsilon = 0$. The underlying reason, however, is that as ϵ approaches 0, the exponential convergence rate constant γ from the Bakry–Emery condition tends to 0, as illustrated in the left panel of Fig. 1.

7.5. Second case study

This section is about a nonlinear stochastic system with two states and one control input, given by

$$f(x) = \begin{pmatrix} x_2 - \frac{1}{2}x_1x_2 \\ -x_1 \end{pmatrix} \quad G(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \epsilon = 0.2,$$

on the domain $\Omega = [-3, 3]^2$ and control constraint set $\mathbb{U} = [-4, 4]$. An associated non-convex stage cost function is given by $\ell(x, u) = \frac{1}{4}x_1^2 + 3(x_2 - 1)^2 + \frac{1}{2}u^2$. This example trivially satisfies Assumptions 1, 2, and 3. Assumption 4 can be verified numerically finding the exponential decay rate $\gamma \approx 0.06$. Fig. 2 visualizes the optimal infinite horizon control law μ_∞ , the optimal steady state ρ_∞ , and the associated dual solution V_∞ that solves the ergodic HJB at $\ell_\infty \approx 2.03$. This solution has been found by using a Galerkin discretization with Lagrange basis functions on a simple equidistant grid with $m = 22500$ basis functions and $N = 10^3$ time iterations with $h = 0.05$. The table below,

m	N	$\ \mu_\infty^m - \mu_\infty^*\ _{L^2(\Omega)}$	CPU – time
100	10^3	0.17	9 ms
400	10^3	0.015	35 ms
900	10^3	0.0056	79 ms

lists the CPU run-time of a JULIA-prototype implementation of (58) for different choices of m . The third column of this table lists the L^2 -norm between the control law μ_∞^m that is found by using m basis functions and the control law μ_∞^* , found by using 22500 basis functions.

8. Conclusions

This paper has presented methods for analyzing ergodic- and infinite horizon stochastic optimal control problems. Theorem 1 has established the existence and uniqueness of ergodic optimal controls under Assumptions 1, 2, and 3. These assumptions are satisfied for most nonlinear stochastic control problems of practical relevance. For example, nonlinear McKean–Vlasov systems on bounded domains with reflecting boundaries always admit a Lyapunov–Has’minskii function.

Moreover, this paper has proposed to introduce a novel class of Lyapunov diffusion operators that can be used to formulate a

generalized Bakry–Emery condition, as summarized in [Assumption 4](#). The relevance of this Lyapunov diffusion operator based LMI has been clarified in [Theorem 2](#), which uses this condition to establish the existence of solutions to ergodic HJB equations for nonlinear optimal control problems with indefinite stage costs.

For the first time, this paper has presented methods for analyzing a very general class of nonlinear HJBs by using functional analysis methods in Hilbert–Sobolev spaces that would usually find applications in the context of linear parabolic PDEs. Although solving linear PDEs in high dimensional state spaces remains computationally demanding, the presented results have far reaching consequences. This is because the proposed operator-theoretic dynamic programming method renders existing numerical methods for discretizing linear parabolic PDEs fully applicable for solving nonlinear HJBs. The exploitation of the implications of this observation could be a fruitful direction for future research on stochastic optimal control.

References

- Arisawa, M. (1998). Ergodic problem for the Hamilton–Jacobi–Bellman equation (II). *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 14, 415–438.
- Arisawa, M., & Lions, P.-L. (1998). On ergodic stochastic control. *Communications in Partial Differential Equations*, 23(11–12), 333–358.
- Arnold, A., Markowich, P., Toscani, G., & Unterreiter, A. (2001). On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker–Planck type equations. *Comm. Partial Differential Equations*, 26(1–2), 43–100.
- Aronna, M. S., & Tröltzsch, F. (2021). First and second order optimality conditions for the control of Fokker–Planck equations. *ESAIM – Control, Optimisation and Calculus of Variations*, 27(15).
- Ašković, V., Trélat, E., & Zidani, H. (2022). On the asymptotic behavior of the value function in large time optimal control problems. *IFAC-PapersOnLine*, 55(16), 38–43.
- Bakry, D., & Émery, M. (1985). Diffusions hypercontractives. *Séminaire de Probabilités, XIX, Lecture Notes in Math.* 1123, 177–206.
- Barbu, V. (2021). The controllability of Fokker–Planck equations with reflecting boundary conditions and controllers in diffusion term. *SIAM Journal on Control and Optimization*, 59(1).
- Barbu, V., & Röckner, M. (2023). The evolution to equilibrium of solutions to nonlinear Fokker–Planck equation. *Indiana University Mathematics Journal*, 72(1), 89–131.
- Bardi, M., & Capuzzo-Dolcetta, I. (1997). *Optimal control and viscosity solutions of Hamilton–Jacobi–Bellman equations*. Birkhäuser.
- Barles, G., & Souganidis, P. E. (2000). On the large-time behavior of solutions of Hamilton–Jacobi equations. *SIAM Journal on Mathematical Analysis*, 31(4), 925–939.
- Bellman, R. E. (1957). *Dynamic programming*. Princeton University Press.
- Bismut, J.-M. (1973). Conjugate convex functions in optimal stochastic control. *Journal of Mathematical Analysis and Applications*, 44, 383–404.
- Bobkov, S. G., Gentil, I., & Ledoux, M. (2001). Hypercontractivity of Hamilton–Jacobi equations. *Journal de Mathématiques Pures Et Appliquées*, 80(7), 669–696.
- Bogachev, V. I., Krylov, N. V., Röckner, M., & Shaposhnikov, S. V. (2015). *Fokker–Planck–Kolmogorov equations*. American Mathematical Society.
- Bolley, F., & Gentil, I. (2010). Phi-entropy inequalities for diffusion semigroups. *Journal de Mathématiques Pures Et Appliquées*, 93, 449–473.
- Breiten, T., Kunisch, K., & Pfeiffer, L. (2018a). Control strategies for the Fokker–Planck equation. *ESAIM. Control, Optimisation and Calculus of Variations*, 24(2), 741–763.
- Breiten, T., Kunisch, K., & Pfeiffer, L. (2018b). Infinite-horizon bilinear optimal control problems: Sensitivity analysis and polynomial feedback laws. *SIAM Journal on Control and Optimization*, 56(5), 3184–3214.
- Brenner, S., & Scott, R. L. (2005). *The mathematical theory of finite element methods*. Springer.
- Chapman, S. (1928). On the Brownian displacements and thermal diffusion of grains suspended in a non-uniform fluid. *Proceedings of the Royal Society (A)*, 119, 34–54.
- Ciotir, I., & Fayad, R. (2022). Nonlinear Fokker–Planck equation with reflecting boundary conditions. *Journal of Differential Equations*, 321(1), 296–317.
- Crandall, M. G., Ishii, H., & Lions, P.-L. (1992). User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American Mathematical Society*, 27(1), 1–67.
- Fleming, W. H., & Soner, H. M. (1993). *Controlled Markov processes and viscosity solutions*. Springer.
- Fleming, W. H., & Vermes, D. (1989). Convex duality approach to the control of diffusions. *SIAM Journal on Control and Optimization*, 27(5), 1136–1155.
- Fokker, A. D. (1914). Die mittlere energie rotierender elektrischer dipole im strahlungsfeld. *Annalen der Physik*, 4(3), 810–820.
- Gross, L. (1975). Logarithmic Sobolev inequalities. *Amer. J. Math.*, 97(4), 1061–1083.
- Has'minskiĭ, R. Z. (1960). Ergodic properties of recurrent diffusion processes and stabilization of the solution of the Cauchy problem for parabolic equations. *Theory of Probability and its Applications*, 5, 179–196.
- Hinze, M., Pinnau, R., Ulbrich, M., & Ulbrich, S. (2009). *Optimization with PDE constraints*. Springer.
- Huang, W., Ji, M., Liu, Z., & Yi, Y. (2015). Steady states of Fokker–Planck equations: I. existence. *Journal of Dynamics and Differential Equations*, 27, 721–742.
- Kolmogoroff, A. N. (1931). Über die analytischen methoden in der Wahrscheinlichkeitsrechnung. *Mathematische Annalen*, 104, 415–458.
- Korda, M., & Mezić, I. (2018). Linear predictors for nonlinear dynamical systems: Koopman operator meets model predictive control. *Automatica*, 93, 149–160.
- Krylov, N. V. (1987). *Nonlinear Elliptic and Parabolic Equations of the Second Order*. Reidel, Dordrecht.
- Krylov, N. V. (2008). *Controlled diffusion processes*. Springer.
- Lasserre, J. B., Henrion, D., Prieur, C., & Trelat, E. (2008). Nonlinear optimal control via occupation measures and LMI relaxations. *SIAM Journal on Control and Optimization*, 47, 1649–1666.
- Ledoux, M. (1992). On an integral criterion for hypercontractivity of diffusion semigroups and extremal functions. *Journal of Functional Analysis*, 105(2), 444–465.
- Namah, G., & Roquejoffre, J. M. (1999). Remarks on the long time behavior of the solutions of Hamilton–Jacobi equations. *Communications on Partial Differential Equations*, 24, 883–893.
- Oksendal, B. (2000). *Stochastic differential equations*. Springer.
- Planck, M. (1917). Über einen Satz der statistischen Dynamik und seine Erweiterung in der Quantentheorie. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 324–341.
- Puterman, M. L. (1977). Optimal control of diffusion processes with reflection. *Journal of Optimization Theory and Applications*, 22(1).
- von Smoluchowski, M. (1915). Über Brownsche Molekularbewegung unter Einwirkung äußerer Kräfte und den Zusammenhang mit der verallgemeinerten Diffusionsgleichung. *Annalen der Physik*, 353(24), 1103–1112.
- Sznitman, A.-S. (1984). Nonlinear reflecting diffusion process, and the propagation of chaos and fluctuations associated. *Journal of Functional Analysis*, 56(3), 311–336.
- Trélat, E., & Zuazua, E. (2015). The turnpike property in finite-dimensional nonlinear optimal control. *Journal of Differential Equations*, 258(1), 81–114.
- Trevisan, D. (2016). Well-posedness of multidimensional diffusion processes. *Electronic Journal of Probability*, 21, 1–41.
- Villanueva, M. E., Jones, C. N., & Houska, B. (2021). Towards global optimal control via Koopman lifts. *Automatica*, 132(109610).
- Vinter, R. (1993). Convex duality and nonlinear optimal control. *SIAM Journal on Control and Optimization*, 31(2), 518–538.
- Wonham, W. M. (1966). Liapunov criteria for weak stochastic stability. *Journal of Differential Equations*, 2, 195–207.



Boris Houska is an associate professor at the School of Information Science and Technology at ShanghaiTech University. He received a diploma in mathematics from the University of Heidelberg in 2007 and a Ph.D. in electrical engineering from KU Leuven in 2011. From 2012 to 2013, he was a postdoctoral researcher at the Centre for Process Systems Engineering at Imperial College London. Boris Houska's research interests include numerical optimization, optimal control, robust and global optimization, and fast model predictive control algorithms.